On the Relation Between Credit Spread Puzzles and the Equity Premium Puzzle[1]

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Abstract

We ask whether the equity premium puzzle and the credit spread puzzle can be simultaneously explained by theoretical models. Specifically, we explore models that have been successful at explaining historical equity returns (e.g., Campbell and Cochrane (CC 1999) and Bansal and Yaron (BY 2004)). We find that large time-varying risk premia are essential for explaining the credit spread puzzle. However, such a feature can generate the counterfactual prediction that forward-looking default rates are pro-cyclical, since in such an economy low expected returns in good times implies a greater probability of reaching the default boundary. Therefore, to match the historical counter-cyclical default rates, we propose a default boundary that is more countercyclical than what can be explained solely with time-varying leverage ratios. Such a model captures the credit spread puzzle, and provides a possible explanation for why macroeconomic factors (e.g., Fama-French factors) can explain the common variation of credit spreads even after controlling for leverage, volatility and term structure factors. We conclude that both the equity premium and the level as well as dynamics of credit spreads can be explained simultaneously by the same pricing kernels, with a time-varying risk premium as an essential component. To investigate the time-series implications of such models, we feed historical consumption innovations into the CC model (with counter-cyclical default boundary) and show that the predicted credit spreads fit both the level and dynamics of historical credit spreads reasonably well.

1 Introduction

It is well-known that standard models of credit risk predict counterfactually low credit spreads for corporate debt, especially for investment grade bonds of short maturity. Early work includes Jones, Mason and Rosenfeld (1984), who find that the Merton (1973) model generates yield spreads that fall far below empirical observation for investment grade firms. Although subsequent work (e.g., Eom, Helwege and Huang (2003)) has found that various structural models can generate very diverse predictions for credit spreads, Huang and Huang (HH 2003) demonstrate that once these various models are calibrated to be consistent with historical default and recovery rates, they all produce similar results. In particular, under this calibration, all structural models produce counterfactually low credit spreads. For example, HH report that the theoretical average 4-year BBB-Treasury spread is 32bp, and relatively stable across models. This contrasts sharply with the historical average BBB-Treasury spread of 158bp. Similarly, HH find that the theoretical average 4-year AAA-Treasury spread is about 1bp, versus the historical average of 55bp.

The typical 'explanation' for the large discrepancy between observed and theoretically predicted spreads is that these theoretical models account only for credit risk. That is, these models choose to ignore other factors that affect corporate bond prices, such as taxes, call/put/conversion options and the lack of liquidity in the corporate bond markets.[1] However, assuming that the component of the credit spread due to these issues is of similar magnitude for AAA and BBB bonds, then the (BBB-AAA) spread should be mostly due to credit risk. Yet, standard structural models of default also fail to generate a satisfactory level for this spread. Indeed, note that the HH results imply a theoretical estimate for the BBB-AAA spread of (32 - 1) = 31bp, which falls far short of the observed spread of (158 - 55) = 103bp. Below, it is this discrepancy between observed and theoretical predictions[2] of the credit spreads that we shall call the credit spread *level puzzle*.

While the *level* of spreads has received considerable attention,[3] below we document that standard pricing kernels also generate counterfactually low predictions for the *time-variation* in credit spreads.[4] For example, a long time series of BBB-AAA spreads generates a population standard deviation of 72bp. In contrast, our benchmark model for a typical refreshed BBB

[1] See Elton et al. (2001), Driessen (2002) for attempts to decompose the spreads in various components.

[2] As pointed out by HH, we emphasize that the theoretical predictions are obtained *conditional on fitting the historical average expected loss.*

[3] See, for example, Jones, Mason and Rosenfeld (1984), HH (2003), Eom, Helwege and Huang (2003), Ericson and Reneby (2003), Driessen (2004)

[4] Papers that look at implications of credit spread changes for structural models are Collin-Dufresne, Goldstein and Martin (2000), and for reduced-form models Berndt, Douglas, Duffie and Schranz (2003).

credit quality firm (where the risk premium, risk free rate, payout rate, firm volatility, and default boundary are all constant, and only initial leverage levels change) generates a predicted time variation of only 38bp. Below we shall refer to this as the credit spread *volatility puzzle.*

Note that in order to identify a credit-risk model that is calibrated to match historical default and recovery rates, we need to specify i) a pricing kernel, ii) firm value dynamics, iii) default boundary dynamics, and iv) recovery rate dynamics. Thus, at first blush there would seem to be considerable flexibility in specifying a model in order to explain the puzzles mentioned above. However, HH find that credit spread predictions are extremely insensitive to the many different variations that researchers have proposed once these models are forced to match historical default and recovery rates. Intuitively, this result can be understood by noting that yield spreads have a one-to-one mapping with (in the example given here, zero coupon) bond prices, and that the latter must, under some relatively weak no-arbitrage restrictions (see, e.g., Cochrane (2001) or Duffie (1996)), satisfy the following relation:

$$\begin{aligned} P &= \mathrm{E}\left[\Lambda\,(1-\mathbf{1}_{\{\tau\le T\}}L_\tau)\right] \\ &= \mathrm{E}\,[\Lambda]\,\mathrm{E}\left[1-\mathbf{1}_{\{\tau\le T\}}L_\tau\right] + \mathrm{Cov}\left[\Lambda\,,(1-\mathbf{1}_{\{\tau\le T\}}L_\tau)\right] \\ &= \frac{1}{R^f}\left(1-E\left[\mathbf{1}_{\{\tau\le T\}}L_\tau\right]\right) - \mathrm{Cov}\left[\Lambda\,,\mathbf{1}_{\{\tau\le T\}}L_\tau\right]. \end{aligned} \tag{1}$$

Here, Λ is the pricing kernel, τ is the time of default, and L_τ is the loss given default. By calibrating expected default and recovery rates, HH force all models to agree on the expected future cash flows $E\left[(1-\mathbf{1}_{\{\tau\le T\}}L_\tau)\right]$ (the first term on the RHS). Hence, in order to predict lower prices for risky bonds (and thus higher spreads) consistent with the historical expected loss rate, we must search for models which:

- increase the covariation between the pricing kernel and the default time,
- increase the covariation between the pricing kernel and the loss given default.

Structural models typically assume that the default event is *value based* (as opposed to liquidity triggered). As a result, default is modeled as the first passage time of the firm's asset value process V_t to some default boundary B_t (which is often related to the outstanding liabilities of the firm):

$$\tau := \inf\{t\ :\ V_t \le B_t\}$$

Thus, in order to predict lower bond prices conditional on a given expected historical loss rate in structural models we have basically three options:

- increase the covariation between the pricing kernel and asset values,

- decrease the covariation between the pricing kernel and the default boundary processes,
- increase the covariation between the pricing kernel and the loss rate.

Below we investigate these possibilities.[5] The above argument suggests that the systematic nature of default events provides the main hope of resolving the credit spread level puzzle for structural form models. This is reminiscent of the well-known 'equity premium puzzle.' As such, this paper investigates whether models that have been successful in explaining historical equity returns (e.g., Campbell and Cochrane (CC 1999) and Bansal and Yaron (BY 2004)) can also explain credit spreads. Such an exercise is meaningful in several respects. First, it can help validate the common empirical practice of linking equity premium to credit spreads (e.g., Chen, Roll, and Ross (1986), Keim and Stambaugh (1986), Campbell (1987), Fama and French (1989, 1993), Ammer and Campbell (1993), and Jagannathan and Wang (1996)). Second, explaining credit spreads provides an 'out-of-sample' test of asset pricing models that are successful at fitting equity market data. It can thus help discriminate between various 'explanations' of the equity premium. Third, besides the credit spread level, which received much attention, we can study the co-variation between equity and macro-economic data and credit spreads. In particular, we focus on the time-series prediction of the model once calibrated to historical macroeconomic data (consumption, price/dividend ratio, etc.) for the dynamics of credit spreads.

As discussed above, both CC and BY successfully capture statistics from equity markets by proposing pricing kernels that covary strongly with consumption. Interestingly, their explanations are quite different. To provide some intuition, recall that the simple discounted dividend growth model predicts that stock value V satisfies $V = \frac{D}{k-g}$, where D is the current dividend rate, k is the appropriate discount rate for this firm's cash flows, and g is the expected growth rate of dividends. CC captures the equity premium puzzle by assuming consumption dynamics that have a constant growth rate g and constant volatility σ, but a strongly countercyclical cost of capital (or equivalently, risk premium) k. In contrast, BY capture the equity premium puzzle by considering two channels: the first channel consists in a constant cost of capital (and hence a constant risk premium) k and small but persistent innovations to the growth rate g. The second channel focuses on time-varying consumption volatility σ which also generates time-varying risk premia. In order to isolate the two effects of the second channel, below we separately investigate models with i) time-varying consumption volatility and a constant risk-premium, and ii) time-varying consumption volatility and a time-varying risk-premium.

[5] There is some evidence in favor of countercyclicality in loss rates for investment grade bonds (e.g., Altman (2000), Altman et al. (2005) and Acharya et al. (2004)).

We estimate credit spreads in both models, with pricing kernels calibrated to equity data, by using a standard structural model (e.g., Black and Cox (1976), Longstaff and Schwartz (1995), Collin-Dufresne and Goldstein (2001)) where firms default the first time their asset value falls below a default boundary. Following HH, the default rates and recovery rates are calibrated to fit the historical data. In addition to the benchmark constant default boundary case as in HH, we also consider stochastic boundaries in order to match historical relations between spreads and expected future default rates.

Our main results are as follows. First, none of the models can explain either the average level or the time-variation of the AAA-Treasury spread. Simply put, the historical default frequencies are too low to be explained from a credit perspective. This result is consistent with interpreting both the level and the time variation of the AAA-Treasury spread to be mostly non-default related.[6] Interestingly, since there is a strong positive correlation between the AAA-Treasury spread and the BBB-AAA spread, this in turn suggests (taking the credit spreads predictions at face value) that liquidity, defined as the non-default component of spreads, moves with the business cycle.[7]

Second, the CC model with a *constant* default boundary can capture a large (but insufficient) portion of the spread level, but this model predicts counterfactual pro-cyclical default probabilities. Interestingly, if we calibrate the model to match the historical relation between spreads and default rates by imposing counter-cyclical default boundary dynamics, then the model can capture all of the average (BBB-AAA) level. In other words, a proper combination of the CC pricing kernel and a counter-cyclical default boundary can capture both the level and the time variation of the BBB over AAA spread. We further show that this default boundary must be more counter-cyclical than naturally implied due to variation in leverage ratios. An interesting implication is that macroeconomic factors that covary with the counter-cyclical default boundary can explain the variation of credit spreads even after controlling for firms characteristics such as leverage ratios. Indeed, this is consistent with the empirical findings of Collin-Dufresne, Goldstein and Martin (2002), Cheyette et al(2003), and Shaefer and Strebulaev (2004) who document that market wide (e.g., Fama-French) factors are economically and statistically significant for predicting changes in credit spreads even after controlling for changes in firm value, volatility, jump probabilities, interest rates, etc.

[6]Several papers have argued that the Treasury rate should not be the right 'risk-free rate' benchmark due to taxes and time-varying liquidity, e.g., Grinblatt (2000), Collin-Dufresne and Solnik (2001), He (2001), Longstaff (2003), Hull and White (2004)

[7]Of course, an alternative explanation is a 'Peso' problem in the bond market, i.e., the fact that the market accounts for the possibility of a so-far unobserved event where many investment grade firms would default jointly.

Third, in both 'BY' models that explain the equity premium solely with cash-flow risk (i.e., where the risk-premium is constant), the model is unable to explain much more of the credit spread than the simple 'benchmark model.' However, in the time-varying risk premium model, significantly more (though still not all) of the observed (BBB - AAA) spread can be captured. This suggest that time varying risk-premia are an essential feature for a pricing kernel to explain both the stock and corporate bond markets.

Finally, we focus on the time series properties of spreads predicted by the two models once calibrated to equity and consumption data. We show that the consumption surplus ratio - the key driver of equity premium in the CC model - built using consumption data from 1919-1997, provides a striking inverse image to the historical credit spreads. The simulated credit spread fits the mean and variation of historical BBB over AAA spread quite well. The simulated and actual credit spreads are 67% correlated for the whole sample period, though the correlation drops in the post war period.

The rest of the paper is as follows. In Section 2, we provide historical data on the level and time variation of credit spreads, leverage and default probabilities. Also, we develop a benchmark. In Section 3, we review the pricing kernel of CC and present its implications for credit spreads. In Section 4, we present a continuous time version of the Bansal-Yaron model and present its implications for credit spreads. We conclude in Section 5. In the Appendix, we review some of the numerical predictions of the CC model.

2 Historical data, summary statistics, and benchmark

In this section, we first present some summary statistics related to macroeconomic variables and default risk. These empirical patterns could provide important yardsticks in gauging the success of model simulations. Second, we develop a standard structural form model and present simulated credit spreads as a benchmark.

2.1 Historical statistics

In Panel A of Table 1, the price dividend ratio (P/D ratio), obtained from John Campbell's website, is on average 23.40 for the 1919-1997 period. The average BBB over AAA spread (from the Federal Reserve Board) is 1.22% for the 1919-1997 period.[8] It is highly volatile with a standard deviation of 0.72%. The 4-year future cumulative default rate (from Moody's annual reports) is on average 1.39% with a standard deviation of 1.08% for the 1970-1998

[8]HH report an average of 158 bp for BBB bonds using a different sample.

period. Noting that the default rate is 4-year cumulative, it appears to be very low compared to the level of credit spread even without considering the effect of recovery rates (i.e., expected four year losses are low compared to spreads). In addition, a regression of the four-year default rate on the BBB over AAA spread yields a highly significant coefficient of 1.2.

To obtain the leverage ratio, we first calculate the market value of debt per dollar of face value for each firm-year (from the Lehman Brothers fixed income dataset). We then scale the book debt (from COMPUSTAT) by this number to obtain the market value of debt. We provide three measures of leverage ratio. The first is the book leverage (BLV), calculated as the ratio of book debt to (book debt + market equity). The second is the market leverage (MLV) as the ratio of market debt to (market debt + market equity). The third is the inverse distance to default (IDD), as the ratio of (0.5*long term book debt + short term book debt) to (market debt + market equity). This last measure is similar to that used by KMV in their implementation of the Black Scholes Merton model to estimate their expected default frequencies (EDF).

All measures cover the 1974-1998 period due to restrictions of the Lehman Brothers fixed income dataset. We only report the leverage ratios of BBB rated bonds. IDD is on average 28%, much lower than BLV (46%) and MLV (44%). We present the correlation matrix in Panel B. In addition to the above variables, we also include consumption innovation, defined as the demeaned log aggregate consumption. The following patterns can be observed. First, the BBB over AAA spread is highly counter-cyclical: it significantly covaries with the P/D ratio and the consumption innovation. In addition, the 4-year future default rate is significantly positively related to BBB over AAA spread, suggesting that credit spread properly incorporates the variation of future default rates. (Although we demonstrate below that the time variation in leverage ratios is not sufficient to explain the time variation in spreads.) Furthermore, the three leverage ratio measures appear to be counter-cyclical because they are significantly negatively related to the P/D ratio and positively related to the BBB over AAA spread. Among the three measures, MLV is the least counter-cyclical because the comovement of both market debt and equity partly offsets each other. On the other hand, IDD is the most counter-cyclical because both market values of debt and equity move with the business cycle given the same book debt. We plot in Figure 2.1 the three leverage ratios of BBB rated bonds as well as BBB over AAA spread for the 1975-1998 period. We define a year as in recession if there are at least five months in that year that are defined as in recession by NBER. It is clear that during the two recession periods the three leverage ratios go up, reflecting the fact that market equity values go down more than debt values and/or firms are not cutting debt levels.

Panel A:	Summary statistics			
Variables	Mean	Std.	Min.	Max.
P/D ratio	23.40	7.22	11.48	49.43
BBB over AAA spread (%)	1.22	0.72	0.37	4.20
4-year default probability (%)	1.39	1.08	0.00	4.00
Book leverage of BBB	0.46	0.09	0.27	0.62
Market leverage of BBB	0.44	0.08	0.29	0.59
Inverse of the DD of BBB	0.28	0.07	0.16	0.42

Panel B:	Correlation matrix of some benchmark variables						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
P/D ratio (1)	1.00						
Consumption innovation (2)	0.11	1.00					
	0.35						
BBB over AAA spread (3)	-0.43	-0.33	1.00				
	0.00	0.00					
BBB 4-year default probability (4)	-0.27	0.02	0.47	1.00			
	0.16	0.93	0.01				
Book leverage of BBB (5)	-0.70	-0.29	0.55	0.12	1.00		
	0.00	0.17	0.01	0.58			
Market leverage of BBB (6)	-0.61	-0.23	0.47	0.07	0.96	1.00	
	0.00	0.27	0.02	0.74	0.00		
Inverse of the DD of BBB (7)	-0.68	-0.39	0.58	0.16	0.96	0.86	1.00
	0.00	0.06	0.00	0.46	0.00	0.00	

Panel C:	Regressions of default probability on BBB - AAA spread		
Dependent variable	intercept	BBB - AAA spread	Adj R sq
4-year default probability	0.02	1.2	0.19
	0.36	4.09	

Table 1: Summary statistics. The statistics of different variables cover different periods. The BBB over AAA spread and the P/D ratio cover 1919-1997 period. The 4-year default probability covers 1970-1998 period. Book leverage is defined as the ratio of book debt to (book debt + market equity). Market leverage is defined as the ratio of market debt to (market debt + market equity). The inverse of the distance to default (DD) is defined as the ratio of (0.5*long term book debt + short term book debt) to (market debt + market equity). In panel B, the first (second) row is the correlation (p-value). The correlation statistics use the maximum common sample size between two series. In Panel C the the first row is the OLS regression coefficients. On the second row Newey-West t-statistics are reported. 4 lags are chosen for 4-year default probability.

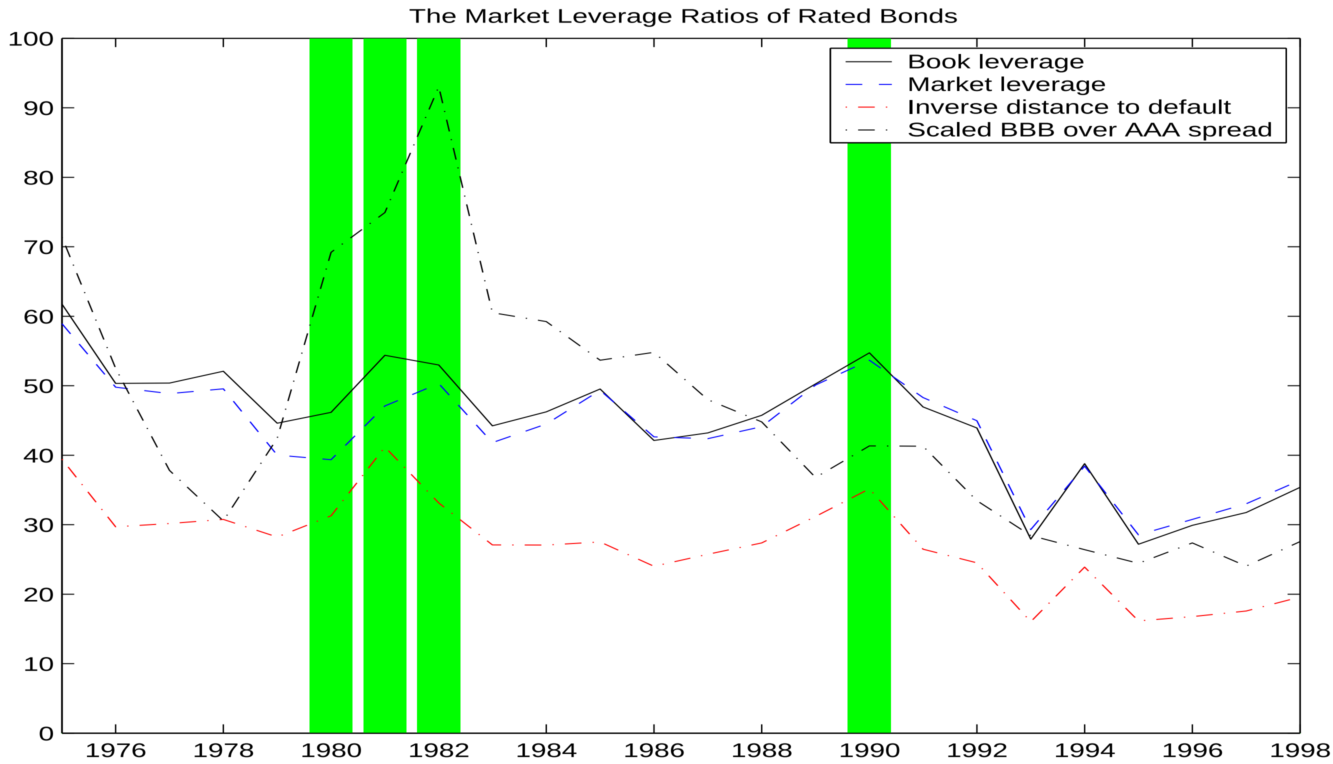

Figure 1: Time series of leverage for BBB rated firms.

To summarize our empirical evidence:

- BBB-AAA spreads are high on average (122 bp) and very volatile (72 bp std dev).
- BBB Default rates are low on average (1.4 percent four year default probabilities) and volatile.
- Default rates are countercyclical in that the correlation between spreads and default rates is positive.
- Finally, the typical BBB firm leverage appears to be counter-cyclical and positively related to credit spread.

Next, we will present the simplest benchmark model (similar to that used by HH) which basically corresponds to the simple Merton (1976) Black and Cox (1976) framework.

2.2 A benchmark model

As a benchmark we use the most standard structural model in the spirit of Merton (1976). We follow the approach of HH by calibrating the model so that it matches the historically observed average loss rate. For the simplest model we assume that the market portfolio has the following return dynamics:

$$\frac{dV(t)}{V(t)} = (\theta + r - \delta)\, dt + \sigma_V dz_V(t), \tag{2}$$

where all coefficients are constant and z_V is a standard Brownian Motion denoting aggregate shocks. We choose the risk-free rate r, the payout ratio δ, the risk-premium θ to match historically estimated counterparts. As emphasized by HH, we note that the results are relatively insensitive to most of these parameters *given the calibration approach used*. Indeed, we assume that the typical BBB firm has a firm value return dynamics given by:

$$\frac{dP(t)}{P(t)} = \frac{dV(t)}{V(t)} + \sigma_{\text{idio}} dz_{\text{idio}}(t) \tag{3}$$

where σ_{idio} is the idiosyncratic risk associated with a typical BBB firm (z_{idio} is a standard Brownian motion independent of Z_V). That idiosyncratic risk is calibrated so as to match the observed unconditional four year average default loss rate.

The model for default is standard: Default is assumed to occur the first time that the firm value P falls below a constant default boundary B which is set equal to 60% of the average leverage ratio in the rating category. Upon recovery the debt-holder receive a constant fraction of face value. The recovery rate is set equal to the average historical recovery 51.31%. Parameters for the market are chosen to match the historical counterparts.

The above calibration is very similar to that proposed in HH and, not surprisingly, our benchmark results are very similar to theirs. Our results differ in one respect. Unlike HH we also look at the implications of this benchmark model for the predicted volatility of spreads. We note that the only source of variation in spreads in this model is the initial distance to default, i.e., $\mathrm{D} = \frac{V}{B}$ (this is similar to Merton (1976)).

Thus, the model predicts a variation in the spread for a typical BBB firm that is a direct function of the variation in the distance to default.

We note that the benchmark model assumes that distance to default is a constant fraction of initial leverage. Historically, we found that the volatility in log-leverage is about 22%. To compute an implied population variance for credit spreads, we simply sample the initial log-leverage ratio from a Gaussian distribution with mean and variance matching their historically observed counterparts. We simulate the model from these different starting values and present sample means and variances.

The parameters used for all simulations are:[9] r =0.039, δ = 0.0523, θ = 0.066, σ_V = 0.19, σ_{idio} =0.203, $\exp E[\log \text{D}]$ = 0.4328, $\sigma(\log \text{D})0.22$ The predictions of the model are presented in table 2.

P def prob	P Std dev	Q def prob	Q std dev	Average Spread	Std Dev. Spread	Reg Coef
0.0141	0.013	0.0383	0.03	0.0046	0.0038	3.44
(0.0001)		(0.0002)				

Table 2: Estimated values of P and Q default probabilities as well as the unconditional mean and variance of the credit spread for four year to maturity BBB firms. Standard Errors of estimates are in parenthesis. Parameters of the typical BBB firm are as defined above for the dividend claim with added 20.3% idiosyncratic volatility. The spread is simulated within a structural model which assumes a constant boundary at 60% of the average BBB leverage ratio ($K = 0.6 * 0.4328$). Upon default bond recover constant fraction of face value corresponding to average historical BBB recovery rate 51.31%. All quantities are in basis points. Simulations are run with 200,000 runs for each price estimation (conditional on state), using a variance reduction technique based on importance sampling discussed in the appendix. Q-default probability and spreads can also be computed using explicit solutions.

The model fails in several respects:

- The average spread is too low ($\approx 45bp$).
- The volatility of the spread is too low. In this model once we control for leverage, spreads are essentially constant (note that it would be zero if the typical BBB firm maintained a constant *initial* leverage through the cycle).
- The covariance between the real default probability and the spread is too high compared to the variance of spread, measured by the regression coefficient of 3.44 (the corresponding historical number is 1.20 as in Table 1).
- The default rates are highly sensitive to the business cycle as measured by spreads (or leverage), but *controlling for leverage* there is no common factors in spreads.[10]

In the next section we investigate how more 'sophisticated' models that are engineered to fit equity data fare in predicting levels and time variation in credit spreads once they are appropriately calibrated to equity data.

[9]We note that the risk-premium used corresponds to equity risk-premium and not to unlevered firm value. However we found that using an unlevered risk-premium of 4.91% had virtually no effect on the predicted spreads, because of the calibration approach used. It essentially resulted in lower fitted idiosyncratic volatility. We keep this specification for consistency with the numbers used for CC and BY further down.

[10]This is of course an inherent factor the one-factor specification of the benchmark model.

3 The CC Habit Formation Model

Slightly modifying their notation, Campbell and Cochrane (1999) specify the utility function of the representative agent in an exchange economy as

$$U(C_t, \widehat{C}_t, t) = e^{-\alpha t} \frac{\left(C - \widehat{C}\right)^{1-\gamma} - 1}{1-\gamma}, \tag{4}$$

where $\widehat{C}$ is an exogenous habit. Since there are no investment opportunities, and since the dividend is perishable, it follows that in equilibrium consumption equals the dividend payment. Further, the pricing kernel is equal to the marginal utility of the representative agent:

$$\begin{aligned}
\Lambda_t &= U_C(C_t, \widehat{C}_t, t) \\
&= e^{-\alpha t} \left(C - \widehat{C}\right)^{-\gamma} \\
&= e^{-\alpha t} \left(\frac{C - \widehat{C}}{C}\right)^{-\gamma} C^{-\gamma} \\
&\equiv e^{-\alpha t} S^{-\gamma} C^{-\gamma} \\
&\equiv e^{-\alpha t} e^{-\gamma s} e^{-\gamma c},
\end{aligned}$$

where

$$\begin{aligned}
S &\equiv \left(\frac{C - \widehat{C}}{C}\right) \\
s &\equiv \log S \\
c &\equiv \log C.
\end{aligned}$$

CC specify the log-consumption process as

$$\Delta c = g\, \Delta t + \sigma\, \Delta z. \tag{5}$$

Finally, CC specify the log surplus consumption ratio dynamics as[11]

$$\Delta s = \begin{cases} \kappa(\overline{s} - s)\Delta t + \sigma \left[\frac{1}{\overline{S}}\sqrt{1 - 2(s - \overline{s})} - 1\right] \Delta z & \text{for} \quad s \leq s_{max} \\ \kappa(\overline{s} - s)\Delta t & \text{for} \quad s > s_{max}, \end{cases}$$

where

$$\overline{S} \equiv \sigma\sqrt{\frac{\gamma}{\kappa}} \tag{6}$$

$$s_{max} \equiv \overline{s} + \frac{1}{2}\left(1 - \overline{S}^2\right). \tag{7}$$

[11]We use the parameter κ instead of $(1 - \phi)$ because κ, which has units of inverse-time, can be easily 'annualized' if first measured using a different frequency. In contrast, annualizing ϕ can only be done approximately.

These parameter choices generate an economy with a constant real rate of interest when $s < s_{max}$:

$$r_f = \alpha + \gamma g - \frac{1}{2}\gamma\kappa. \tag{8}$$

The price-consumption ratio for the claim to consumption can be written as

$$\left(\frac{P(t)}{C(t)}\right) = \mathrm{E}_t\left[\frac{\Lambda(t+1)}{\Lambda(t)}\frac{C(t+1)}{C(t)}\left(1+\frac{P(t+1)}{C(t+1)}\right)\right] \tag{9}$$

$$= \mathrm{E}_t\left[\sum_{j=1}^{\infty}\frac{\Lambda(t+j)}{\Lambda(t)}\frac{C(t+j)}{C(t)}\right]. \tag{10}$$

While their framework does not provide analytic solutions for the price-consumption ratio, equations (9) and (10) suggest two numerical schemes for estimating this ratio. In particular, equation (9) can be estimated by using a recursive scheme to obtain a self-consistent solution for $\frac{P}{C}$. Alternatively, equation (10) can be estimated using Monte-Carlo methods. Unfortunately, both methods are vulnerable to certain types of errors, as discussed in the Appendix.[12] Indeed, there we demonstrate that their estimated price-consumption ratio (which generates all of their later results) differs significantly from our estimate.

The habit formation model of CC makes an ambitious attempt to fit post-war consumption and security pricing data using a model with only two free parameters: γ and κ. In their model, γ is chosen to match the unconditional Sharpe ratio, while κ is chosen to match the serial correlation of the log price-dividend ratio. Once these two parameters are specified, the remaining moments of historical data are implied by their model. Three of the most important 'out of sample' predictions that CC claim to capture within their model are 1) mean returns, 2) standard deviation of returns, and 3) the average price-dividend ratio.[13] However, our analysis indicates that their model generates expected returns and standard deviations that are only about one-half of historical levels, while generating average price-dividend ratios that are about twice that of historical levels. We report in Appendix A the simulated statistics.

Separately, we also investigate whether the CC model can capture historical returns when calibrated to fit historical *dividend* (vs. consumption) data. As shown in Appendix A, we find that we can fit the dividend claim reasonably well if γ is increased from 2 to 6. Fortunately, for our purposes, this fitting of the dividend claim is sufficient, as our immediate goal is to explain the time series of credit spreads.

[12] A third approach based on the continuous time version of the model, and which relies on solving a partial differential equation, is discussed in the appendix as well.

[13] Note that since the Sharpe ratio is a ratio of mean return to standard deviation, and since the model is calibrated to fit the Sharpe ratio, fitting the average historical return guarantees that the average volatility is captured.

3.1 Estimating Credit spreads in the CC Framework

Here we calibrate the CC model to provide estimates for the 4-year credit spread as a function of the state variable s. In addition, we calculate the steady state distribution of s, which in turn generates the average spread level and the population variance.

We perform this calibration by using the CC model to generate the dynamics of the market portfolio:[14]

$$\frac{dV(t)}{V(t)} = \Big(\theta(s_t) + r - \delta(s_t)\Big)\, dt + \sigma(s_t)\, dz_V(t). \tag{11}$$

Here, the risk-free rate is constant, but, unlike in the benchmark model, the risk-premium $\theta(s)$, the dividend yield $\delta(s)$, and volatility $\sigma(s)$ are all stochastic. They are function of the surplus consumption ratio state variable s_t. The exact dependence is uniquely specified by the CC pricing kernel (we obtain them via simulation explained in the appendix). We then assume that the typical firm follows the dynamics

$$\frac{dP(t)}{P(t)} = \frac{dV(t)}{V(t)} + \sigma_{idio}\, dz_{idio}(t). \tag{12}$$

Without loss of generality, we set initial firm value $P(0) = 1$.

3.1.1 Constant default boundary case

Following HH, we set the nominal default boundary to be a constant: $P^{nom}_{def}(s) = 0.6 * 0.4328$. The 0.4328 comes from the average leverage ratio for BBB used by HH, and the (.6) accounts for the fact that firm value can drop well below initial book value of debt before defaulting. This number is consistent with recovery rates of approximately 50%, and bankruptcy costs of approximately 15%. Consistent with the findings of HH, the credit spread estimates are very robust to changes in these estimates, since in order to match historical default rates, a higher boundary, for example, just implies a lower volatility, which tends to cancel any effect on credit spreads. As noted in the introduction, only changing the covariance of the pricing kernel with default and recovery rates will produce significantly different results.

Since the CC model is calibrated in real terms, and since corporate bonds are written in nominal terms, we need to account for these correctly. For simplicity, we assume a constant inflation rate of 2%. Hence, while the nominal default boundary is a constant, the real default boundary decays over time as $P^{real}_{def}(s) = 0.6 * 0.4328e^{-infl*t}$. Coupon rates are set equal to

[14] We note that the implied dynamics of the market portfolio are actually calibrated to match the dynamics of aggregate equity prices. To be fully in line with the structural model we should use unlevered firm value dynamics. However, as discussed for the benchmark model, this effect is relatively minor since our calibration solves for the idiosyncratic level to match expected default rates.

$s(0)$	Steady State Distribution	BBB-Treasury Spread	Q-Default Rate	P-Default Rate
-3.31	0.01	0.0144	0.064	0.010
-2.93	0.06	0.0062	0.054	0.011
-2.56	0.28	0.0059	0.051	0.012
-2.19	0.59	0.0053	0.046	0.015
-1.82	0.07	0.0041	0.036	0.023

Table 3: Model generated BBB state-dependent credit spreads when the nominal default boundary is a constant (equal to 0.6*0.4328). The idiosyncratic risk needed to match historical default rate of 1.41% is $\sigma_{idio} = 0.257$. The unconditional average BBB-AAA credit spread and standard deviation are (53 ± 10)bp, compared to the historical average of (122 ± 72)bp.

the sum of three terms: the constant real risk free rate[15] plus inflation plus 100 basis points. The first two terms sum to give a constant nominal risk free rate. The third term is included so that the risky bond is issued near par. Again, our findings are insensitive to changes in this prescription.

We match the historical 4-year default frequency of 1.41% for BBB bonds. This is accomplished by trial and error - that is, for each different guessed value of σ_{idio}, we estimate the 4-year expected default rate for a range of values for $s(0)$, and then average over the population density. We find that $\sigma_{idio} = .257$ in equation (12) above satisfies such a condition.

With the historical measure fully calibrated to equity data and σ_{idio} calibrated to fit historical default rates, we can generate the risk-neutral dynamics via:

$$\frac{dV(t)}{V(t)} = (r - \delta(s_t))\ dt + \sigma(s_t)\, dz_V^Q(t). \tag{13}$$

Here, the risk-free rate r, dividend yield $\delta(s)$, and volatility $\sigma(s)$ are again pinned down uniquely by the CC model. In turn, the risk-neutral dynamics for a typical firm follow:

$$\frac{dP(t)}{P(t)} = \frac{dV(t)}{V(t)} + \sigma_{idio}\, dz_{idio}(t). \tag{14}$$

Upon default, we assume that the agent receives a recovery of 0.5131, consistent with the recovery rate used in HH. All future promised coupon payments receive zero recovery. We then estimate the BBB-Treasury spread as a function of $s(0)$. The results are tabulated in Table 3.

First note that the model generates an average spread of 53bp with a standard deviation of 10bp. We note that the standard deviation is obtained for a *Constant* initial leverage ratio

[15] recall that $r(s)$ is a constant for all values of $s < s_{max}$, and that, in discrete time, s can actually be greater than s_{max}. Hence, $r(s)$ is stochastic in the CC framework. However, in the continuous time version of the CC model discussed in the appendix, interest rates are truly constant since s_{max} constitutes a natural reflecting boundary.

for which the standard model predicts zero variation. The CC model introduces two elements relative to the benchmark model. First, it introduces time-variation in risk-premia. Second, it adds time variation in payout ratio and volatility of firm value. The latter affects the risk-neutral dynamics of firm value and explains why, even conditional on a fixed initial leverage, the CC model generates time variation in spreads. The former affects the physical measure dynamics. In particular, the average P-measure default probability is 1.41%, the same as that in the benchmark case, but the Q-measure probability is higher than in the benchmark case. Effectively, the counter-cyclical risk premia creates a larger wedge between P-measure and Q-measure default probabilities, as can be seen by comparing Tables 2 and 3.

Further, the simulated average credit spread suggests that the covariance between the state of the economy and default events is still well below what it takes to explain the level observed in spreads. In fact, this model generates the counterfactual prediction that expected default rates are highest in the *best* states of the economy.[16] This occurs because the mean reversion in the dynamics of s implies that a high value of s today implies a lower expected value of s over the next four years. Hence, aggregate market performance $\frac{dV}{V}$, and thus individual firm performance $\frac{dP(t)}{P(t)} = \frac{dV(t)}{V(t)} + \sigma_{idio} \, dz_{idio}(t)$, is expected to be poor over the next few years.

Of course, the above argument is strongly dependent on the initial leverage ratio (i.e., the initial default boundary) being constant across good and bad states. This suggests that, in order to increase the simulated credit spread, we might introduce a counter-cyclical default boundary. Our earlier summary statistics indicate that both default probability and leverage ratios are empirically counter-cyclical. This implies two ways to generate the counter-cyclical default boundary. First, we can define the default boundary as a function of the state (s_t) by empirically regressing default probability on the consumption surplus ratio. Second, we can define the default boundary by assuming that the boundary variation is purely driven by leverage ratio changes, by empirically regressing the leverage ratio on the consumption surplus ratio. In the following we will adopt the first approach. We shall show then that that the second approach is not sufficient to generate the observed credit spreads, and create counter-intuitive pro-cyclical real default rates.

3.1.2 Counter-cyclical default boundary case

As shown above the constant default boundary assumption generates the counterfactual prediction that default rates are pro-cyclical. Instead in this section we consider a simple counter-

[16] Even so, credit spreads in the CC model are still countercyclical, due to the substantial change in risk aversion over the business cycle.

cyclical nominal default boundary given by:[17]

$$P_{def}^{nom}(s) = .4328 - 1.6 * e^{s}. \tag{15}$$

Hence, the real default boundary is specified as

$$P_{def}^{nom}(s) = (.4328 - 1.6 * e^{s})\, e^{-infl*t}, \tag{16}$$

with $infl = 0.02$. We choose the boundary specification (in particular, the 1.6) to closely match the regression coefficient of 4-year default rates on the consumption surplus ratio. The .4328 comes from the average leverage ratio in HH. The argument is that, even in the worst states of nature, the firm value still needs to fall from one to .4328 to default. Also, we note that at s_{max} the location of the default boundary is approximately $P_{def}^{nom}(s_{max}) = 0.22$. Thus the default boundary values with this specification fall with the range $[0.22, 0.4828]$. We interpret this result as follows: at date-0, the leverage ratio is approximately .4328. Historical recovery rates are approximately 50%, which would therefore imply a recovery rate of $(.5) \times 0.4328 = 0.2164$. Hence, even if default occurs in the best states of the economy, there is some positive bankruptcy cost (equal to $0.22 - 0.2164$).

We note that while the specification of the default boundary above may appear somewhat *ad hoc*, it is just as arbitrary to assume that it should be constant! In the end, we need a theoretical model to assess what is a reasonable specification for the boundary process. At present we simply postulate simple boundary dynamics that seem roughly consistent with empirical evidence.

Except for the default boundary, the other specifications are similar to the case of constant default boundary. Following HH, we again choose σ_{idio} so that the average 4-year BBB default rate matches the historical average of 1.41%. For our given specification, we find that $\sigma_{idio} = 0.231$. We then estimate the BBB-Treasury spread as a function of $s(0)$. The results are tabulated in Table 4.

We then repeat the same process for 4-year AAA bonds. Here, we model the nominal default boundary as

$$P_{def}(s) = .1308 - .4 * e^{s}. \tag{17}$$

Again, the .1308 comes from the HH estimate for the average leverage of AAA firms. In order to match the historical AAA 4-year default rate, we choose $\sigma_{idio} = .30$. The results are tabulated in Table 5.

[17] We discuss the empirical implications for leverage and other possible proxies below.

$s(0)$	Steady State Distribution	BBB-treasury Spread	Q-default Rate	P-default Rate
-3.31	0.01	0.0167	0.1281	0.0131
-2.93	0.06	0.0105	0.0822	0.0130
-2.56	0.28	0.0076	0.0598	0.0131
-2.19	0.59	0.0057	0.0448	0.0145
-1.82	0.07	0.0034	0.0268	0.0160

Table 4: Model generated BBB state-dependent credit spreads

s(0)	Steady State Distribution	AAA-Treasury Spread	Q-Default Rate	P-Default Rate+-
-3.31	0.01	0.0010	0.0086	0.0007
-2.93	0.06	0.0007	0.0072	0.0005
-2.56	0.28	0.0006	0.0057	0.0005
-2.19	0.59	0.0004	0.0035	0.0003
-1.82	0.07	0.0001	0.0010	0.0003

Table 5: Model generated AAA state-dependent credit spreads

Finally, we obtain the $BBB - AAA$ spread from Tables (4) and (5). The results are tabulated in Table (6). From these spreads, we obtain a population-average spread of 110bp, with a population standard deviation of 42bp. The level of spread agrees quite well with the historical average of 122bp. The population standard deviation of 42bp is much lower than that of 72 bp obtained for the 1919-1997 period (where much of the volatility comes from the great depression period), but actually similar to the 54bp obtained for the post war period.

In addition, we find that our specification generates a coefficient of 1.297 when regressing the 4-year real default rate on the BBB-AAA spread, well in line with the corresponding coefficient of 1.20 in Table 1. However, our results cannot explain either the average level or the time variation of the AAA-Treasury spread. Taking at face value the prediction of the model, this seems to suggest (very much in line with HH) that much of this spread is due to factors independent of credit risk. The bottom line is that the model can successfully capture the BBB-AAA spread if the default boundary is sufficiently counter-cyclical, a conclusion that is different from HH. The intuition is that, once we match the historical real default rate, the only way to generate a high credit spread is to increase the covariance between the stochastic discount factor and the default risk. While the CC model can induce highly counter-cyclical equity premium, it can also produce counterfactual pro-cyclical real default rates if the default boundary is not counter-cyclical. This is because a higher risk premium during economic downturns means that the P-measure asset value is expected to grow faster and thus can result in lower default rate. A counter-cyclical default boundary mitigates this problem by ensuring that the real default rate is higher during economic downturns, and thus generating credit spread and real default rates consistent with historical evidence. Therefore, the CC model - a model that well explains the equity premium puzzle - can also explain the level and variation of credit spreads if the default boundary is properly specified.

3.1.3 Counter-cyclical default boundary due to leverage changes

It remains to see whether we can tie the variation of the implied default boundary of the previous section to variation in empirical leverage ratios, as suggested in Table 1. To tackle this issue, we first regress the market leverage ratio (MLV) of BBB rated bonds on the exponential surplus consumption ratio and obtain the follow relation:

$$MLV_{BBB}(s) = 0.5210 - 0.6164 * e^{s}. \tag{18}$$

In other words, the leverage ratio is counter-cyclical for a bond that is always rated BBB. Following this relation, we assume that the default boundary is

$$P_{def}^{nom}(s) = (0.5210 - 0.6164 * e^{s}) * 0.6 * e^{-\text{infl}*t}. \tag{19}$$

The rest of the specification, including the inflation rate and the coupon rate, is as before. We find that an idiosyncratic volatility of 0.291 is needed to fit the historical default rate of 1.41%. The simulated results are reported in Table 7. There are several patterns worth noting in the table. First, the average BBB over AAA spread (when combined with the AAA spread result in Table 5) is 60bp, far lower than the 110 bp in the counter-cyclical default boundary case. In other words, a default boundary that is purely driven by leverage ratio changes will not be sufficient to generate the observed spreads. Second, we note that the P-measure default rate in Table 7 is an increasing function of s: the real default rate is pro-cyclical, contradicting to the empirical evidence. In sum, we have shown that the default boundary solely driven by leverage ratio changes will generate too low credit spreads and counterfactual default rates. This suggests that the default boundary must be more countercyclical than what is generated purely from change in leverage ratios. This result has another interesting implication. It is widely documented in the current literature (e.g., Collin-Dufresne, Goldstein, and Martin (2002), Cheyette et al. (2003), Shaefer and Strebulaev (2004)) that aggregate factors (such as the Fama-French risk factors) are economically and statistically significant in predicting changes in credit spreads even after controlling for changes in firms characteristics such as leverage ratio. Our result suggests that these factors may proxy for changes in the default boundary not explained by changes in firm characteristics such as leverage.

s(0)	Steady State Distribution	BBB-AAA Spread
-3.31	0.01	0.0229
-2.93	0.06	0.0193
-2.56	0.28	0.0151
-2.19	0.59	0.0091
-1.82	0.07	0.0023

Table 6: Model generated BBB-AAA state-dependent credit spreads

4 The Bansal-Yaron long-run risk model

In this section we consider the implication for credit spreads of an alternative model, that of Bansal and Yaron (BY 2004). BY's model is very successful in explaining many features of equity data, and in particular: the average equity premium and its volatility, the average price dividend ratio and its volatility, the average risk-free rate and its volatility. In contrast to CC's model, which explains all these features, with iid consumption but time varying risk-aversion generated by the habit process, BY's model has standard (Epstein-Zin type) utility function with constant risk-aversion but modifies the consumption process. It allows both its growth and volatility to follow highly persistent mean-reverting stochastic processes. BY argue that in finite sample their consumption process cannot statistically be distinguished from the iid consumption process assumed in CC. However, it helps explain the equity premium puzzle using a very different mechanism than CC. It is purely based on consumption/cash-flow risk as opposed to the risk-premium/discount rate risk in CC. Implementing the BY model and studying its implications for spreads is thus interesting for two reasons. First, it provides a potential alternative explanation for credit spread level and variation. The results of the calibration can help us sort out which components are more important for spreads. Second, looking at the implications of this model for credit spreads, provides an out-of-sample test of the two explanations of the equity premium: cash-flow risk versus risk-aversion. It seems natural to expect that a model that can explain many features of equity prices should also be able to explain corporate bond prices accurately. A failure along that dimension might indicate that the model is misspecified or that bond and equity markets are segmented. Admittedly both models are highly stylized and may illustrate two different mechanism both of which may be at work in the data. In fact, our results show that time varying cash flow risk alone cannot match either level or time-series properties of spreads. It generates too low average spread level with too high a covariance with default probabilities. Time varying risk-premia appears thus crucial to explain spreads. On the other hand, the results suggest that adding stochastic volatility to a model with time varying risk-aversion may drastically improve the time series properties of predicted spreads (time varying risk premia raise allow for a larger wedge between P and Q measure default probability; stochastic volatility induces counter-cyclical default probabilities).

We first describe the continuous time version of the BY model we implement, then the calibration and the results.

4.1 The 'BY' model

BY propose a complex model based on Epstein-Zin preferences and a consumption process with mean reverting consumption growth and volatility. To solve their model they use several approximations. First, they use a log-linearization of gross returns effectively twice: (i) to express the log pricing kernel as an affine function of the state variables, and (ii) to obtain an affine log price/dividend ratio (for both the consumption and dividend claims). Second, they assume that the variance of log consumption is normally distributed (i.e., possibly negative). Below we propose a continuous time version of their model with similar state variables dynamics (we choose affine dynamics for the consumption, its growth and volatility to match unconditional and conditional moments of BY's model, keeping a positive variance). We follow BY in approximating the pricing kernel as affine,[18] but derive explicit solutions for the price of the consumption and dividend claims (i.e., we do not use the log-linearization approximation to solve for price/dividend ratios). As we document below, the model essentially matches most of the empirical facts of dividend claim as shown in BY. (We also report the numbers for the consumption claim.) So, in principle, the BY model could perform as well as the CC model in explaining spreads once calibrated to successfully fit equity returns. We turn to this in the next section.

Our version of the 'BY' model is:

$$dc_t = (\mu + x_t)dt + (v_t + \bar{v})\, dZ_c(t) \quad (20)$$

$$dd_t = (\mu_d + \phi x_t)dt + \sigma_d(v_t + \bar{v})\, dZ_c(t) \quad (21)$$

$$dx_t = -\kappa x_t dt + \sigma_x(v_t + \bar{v})\, dZ_x(t) \quad (22)$$

$$dv_t = \nu(\bar{v} - v_t)dt + \sigma_v\, dZ_v(t) \quad (23)$$

where c, d are the log consumption and dividend process respectively. Further the pricing kernel is given by:

$$\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - (\lambda_{c0} + \lambda_{c1} v_t)dZ_c(t) - (\lambda_{v0} + \lambda_{v1} v_t)dZ_v(t) - (\lambda_{x0} + \lambda_{x1} v_t)dZ_x(t) \quad (24)$$

[18]We use the approach proposed in Collin-Dufresne and Goldstein (2005) to improve upon the log-linearization of Campbell and Shiller to solve the model.

$$r_t = \alpha_0 + \alpha_x x_t \tag{25}$$

These equation can be compared with equation (A1) and (A10) in the appendix of BY. The two models are identical for the case where volatility is constant and equal to its long-term mean (Case I in BY), and differ only slightly in the case where volatility is stochastic (Case II in BY).[19] We note that one of BY's main contributions is to explicitly relate the parameters of the affine pricing kernel (α_i, λ_j) to preference parameters obtained in an economy where the representative agent has Epstein-Zin preferences and show that for economically plausible preference parameters the above model can explain the equity premium. For our purposes however, we will treat these as free parameters and calibrate them to fit equity data.[20]

In this model we can solve explicitly for the price dividend ratio and all relevant quantities. Under the risk-neutral measure the processes are given by:

$$dc_t = (\mu + x_t)dt + (v_t + \bar{v}) \left(dZ_c^Q(t) - (\lambda_{c0} + \lambda_{c1} v_t)dt\right) \tag{26}$$

$$dd_t = (\mu_d + \phi x_t)dt + \sigma_d (v_t + \bar{v}) \left(dZ_c^Q(t) - (\lambda_{c0} + \lambda_{c1} v_t)dt\right) \tag{27}$$

$$dx_t = -\kappa x_t dt + \sigma_x (v_t + \bar{v}) \left(dZ_x^Q(t) - (\lambda_{x0} + \lambda_{x1} v_t)dt\right) \tag{28}$$

$$dv(t) = \nu(\bar{v} - v(t))dt + \sigma_v \left(dZ_v^Q(t) - (\lambda_{v0} + \lambda_{v1} v_t)dt\right) \tag{29}$$

We first derive the price of the dividend claim which solves:

$$P^d(t) = \mathrm{E}_t^Q\left[\int_t^\infty e^{-\int_t^s r(u)du + d(s)} ds\right] \tag{30}$$

$$= D(t) \int_t^\infty \Psi_{a,b,c,d}(\tau, x_t, v_t) d\tau \tag{31}$$

where we have defined:

$$\Psi_{a,b,c,d}(T-t, x_t, v_t) = \mathrm{E}_t^Q\left[e^{-\int_t^T (a + b x_s + c v_s + d v_s^2) ds}\right] \tag{32}$$

and the constants:

$$a = \alpha_0 - \mu_d + \sigma_d \bar{v} \lambda_{c0} - \frac{1}{2}\sigma_d^2 \bar{v}^2 \tag{33}$$

$$b = \alpha_x - \phi \tag{34}$$

$$c = \sigma_d * (\lambda_{c0} + \lambda_{c1} * \bar{v}) - \sigma_d^2 * \bar{v} \tag{35}$$

$$d = \sigma_d * \lambda_{c1} - \sigma_d^2/2 \tag{36}$$

[19] The only difference between our model relative to case II in BY is that we assume that volatility of consumption growth and (not variance as in BY) follows a Gaussian AR1 process. This avoids the issue of negative variances.

[20] In a separate paper (Collin-Dufresne and Goldstein (2005)) we show that in continuous time the equilibrium prices in an economy where the representative agent has Epstein-Zin utility, can very accurately be represented by an affine economy of the type presented here. In other words, we show that prices obtained in the BY economy without resorting to the Campbell-Shiller log-linearization approximation resemble those obtained in an affine economy (for which we have explicit results).

We find:

$$\Psi_{a,b,c,d}(\tau, x, v) = \exp(A(\tau) + B(\tau)x + C(\tau)v + D(\tau)v^2) \tag{37}$$

where A, B, C, D satisfy a standard system of ODE (the approach follows Duffie and Kan (1996)) that can be solved in closed-form partially and numerically using standard solvers (e.g., Mathematica). For simplicity we do not report the system of ODE.[21]

Note that the consumption claim is obtained similarly

$$P^c(t) = \mathrm{E}_t^Q[\int_t^\infty e^{-\int_t^s r(u)du + c(s)} ds] \tag{38}$$

$$= C(t) \int_t^\infty \Psi_{a',b',c',d'}(\tau, x_t, v_t) d\tau \tag{39}$$

where a', b', c', d' are obtained from a, b, c, d with $\mu_d = \mu_c, \phi = 1, \sigma_d = 1$.

Finally we can get all the quantities we need for calibration, using a two step procedure. First we compute all the conditional moments, then we integrate the conditional moments with respect to the unconditional distribution of the state variables. We obtain the unconditional distribution of x and v by simulation. (We denote by $f(x, v)$ the corresponding joint density). The unconditional moments we compute and report are thus

1. Unconditional mean of log price-dividend ratio:

$$E[\log \frac{P^d}{D}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \log Y(x, v) f(x, v) dx dv \tag{40}$$

2. Unconditional variance of log price-dividend ratio:

$$V[\log \frac{P^d}{D}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\log Y(x, v) - E[\log \frac{P^d}{D}] \right)^2 f(x, v) dx dv \tag{41}$$

3. Unconditional average volatility of log price dividend ratio:

$$E[\left(d \log \frac{P^d}{D} \right)^2]/dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left((\frac{Y_x}{Y})^2 \sigma_x^2 \bar{v}^2 + (\frac{Y_v}{Y})^2 \sigma_v^2 \right) f(x, v) dx dv \tag{42}$$

4. Unconditional average Risk-premium:

$$E[\frac{dP^d + Ddt}{P^d} - rdt]/dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\lambda_{c0} \sigma_d v^2 + \frac{Y_x}{Y} \lambda_{x0} \sigma_x \bar{v} + \frac{Y_v}{Y} \lambda_{v0} \sigma_v v \right) f(x, v) dx dv \tag{43}$$

[21] They are available upon request.

5. Unconditional average volatility of excess return:

$$E[\left(\frac{dP^d + Ddt}{P^d} - rdt\right)^2]/dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\sigma_d^2 v^2 + (\frac{Y_x}{Y})^2\sigma_x^2\bar{v}^2 + (\frac{Y_v}{Y})^2\sigma_v^2\right) f(x,v)dxdv \tag{44}$$

6. Sharpe ratio: divide 4 by 5.

7. Unconditional mean short rate:

$$\mathrm{E}[r] = \alpha_0 + \alpha_x \mathrm{E}[x] \tag{45}$$

8. Unconditional variance of short rate:

$$V[r] = \alpha_x^2 V[x] \tag{46}$$

In the above I have defined $Y = \frac{P}{D} = \int_t^{\infty} \Psi_{a,b,c}(\tau, x, v)d\tau$ so that

$$Y_x = \int_t^{\infty} B(\tau)\Psi_{a,b,c}(\tau, x, v)d\tau \tag{47}$$

$$Y_v = \int_t^{\infty} (C(\tau) + 2vD(\tau))\,\Psi_{a,b,c}(\tau, x, v)d\tau \tag{48}$$

for appropriate parameters a, b, c. Similarly, we obtain the equivalent quantities for the consumption claim. Following BY's table IV we obtain the following parameters:

$$\mu = \mu_d = 0.0015 * 12 \tag{49}$$

$$\kappa = (1 - 0.979) * 12 \tag{50}$$

$$\bar{v} = 0.0263 \tag{51}$$

$$\phi = 3 \tag{52}$$

$$\sigma_x = 0.044 * 12 \tag{53}$$

$$\sigma_d = 4.5 \tag{54}$$

$$\nu = (1 - 0.987) * 6 \tag{55}$$

$$\sigma_v = 0.00177 \tag{56}$$

$$\alpha_0 = 0.0086 \tag{57}$$

$$\alpha_x = 0.48 \tag{58}$$

$$\lambda_{c0} = 0.144 \tag{59}$$

$$\lambda_{x0} = 0.4 \tag{60}$$

$$\lambda_{v0} = -0.3 \tag{61}$$
$$\lambda_{c1} = \lambda_{c0}/\bar{v} = 5.5 \tag{62}$$
$$\lambda_{x1} = \lambda_{x0}/\bar{v} \tag{63}$$
$$\lambda_{v1} = \lambda_{v0}/\bar{v} \tag{64}$$
$$\tag{65}$$

Following BY, λ_{c1} can be interpreted as the relative risk-aversion coefficient and $1/\alpha_x$ can be interpreted as intertemporal rate of substitution.[22] We set the parameters of the consumption and dividend process identical to those of BY except for the volatility parameters, since in our case volatility follows an AR1 process, whereas in BY variance does. We choose the parameters of the volatility process so that the unconditional means and variances of the variance process in our model coincide both under physical and risk-neutral measures with that of BY's model. For the risk-premia parameters we choose parameters of the same order of magnitude as BY in order to match historical statistics on equity returns (we have a few more degrees of freedoms since BY have only 3 free preference parameters - time preference, risk-aversion, intertemporal rate of substitution, whereas we have five $\lambda_{c0}, \lambda_{x0}, \lambda_{v0}, \alpha_0, \alpha_x$).

With these parameters set, we obtain the following results:

	Our BY		**Original BY** $\gamma = 10$		**Our estimates CC:** $\gamma = 6$		**Historical Data**	
Statistics	C-claim	D-claim	C-claim	D-claim	C-claim	D-claim	Postwar data	Long data
E[dc]	1.8	1.8	1.8	1.8	1.88	2.03	1.89	1.72
sigma(dc)	2.7	12.16	2.7	12.16	1.24	9.01	1.22	3.32
E[rf]	0.94	0.94	0.93	0.93	0.94	0.94	0.94	2.92
sigma[rf]	0.97	0.97	0.57	0.57	0.	0.	0.97	?
E(r-rf)/sigma(r-rf)	0.28	0.40	NA	0.37	0.94	0.46	0.43	0.22
E(R-Rf)/sigma(R-Rf)		0.40	NA	0.37	0.94	0.5	0.49	0.3
E(r-rf)		6.7	NA	6.84	5.29	5.93	6.69	3.9
sigma(r-rf)		13.0	NA	18	5.63	12.97	15.73	17.96
exp[E(p-d)]		24.36	NA	19.98	23.03	20.72	24.7	21.16
sigma(p-d)		0.24	NA	0.21	0.08	0.1	0.26	0.27
sigma(kernel)		0.56	0.73	0.73				

Table 7: Means and standard deviations of simulated and historical data.

We see that overall our results agree quite well with the predictions of BY. Differences

[22] Collin-Dufresne and Goldstein (2005) in fact show that this holds in the continuous time version of their economy without performing any log-linearization approximation.

may be due to the additional degree of freedom noted above, the small differences in model specification, or to the fact that our results rely on exact closed form prices within an affine pricing kernel framework, whereas BY's solution seem to correspond to affine approximations to the log price/dividend ratio with the affine kernel. In any case, we start from this calibration to determine the corresponding implications for spreads.

4.2 Implications for Credit Spreads

As before we suppose that the return generating process is that of an average firm with additional idiosyncratic risk. We follow exactly the same procedure as described above for the CC model. We price nominal bonds assuming that inflation is constant and equal to 2%. We find that the level of idiosyncratic risk needed to match the unconditional average four year default probability is $sig_{\text{idio}} = 0.233$.

Further, we note that while the emphasis of BY is on cash-flow risk from two sources (growth rate risk and volatility risk), their model also exhibits time variation in risk-premia. Our parametrization allows us to analyze the implications of each component separately. In particular, to distinguish the impact of growth rate risk, volatility and risk-premia on predicted spreads, we distinguish three cases:

- Case I: growth rate risk (i.e., we set the volatility and risk-premia to be constant: $\sigma_v = 0, v_0 = \bar{v}$ and $\lambda_{j1} = 0$, $j = c, v, x$).
- Case II: growth rate and volatility risk (i.e., we set the risk-premia to be constant: $\lambda_{j1} = 0$, $j = c, v, x$)).
- Case III: growth rate and volatility risk as well as time varying risk-premia.

We keep the same level of idiosyncratic volatility across the three cases for comparison purposes ($sig_{\text{idio}} = 0.233$). The prediction of a constant (nominal) default Barrier Black-Cox (1976), Longstaff and Schwartz (1995) model are given in table 8 below.

The table clearly illustrates the implications of cash flow risk for credit spreads. Cash-flow risk alone generated by time varying growth rate risk without time variation in risk-premia (which also corresponds to case I in BY) cannot explain the spread puzzle. The results in that case (presented in the first row) are only marginally different from our benchmark constant coefficient case presented in table (2).

We note that relative to the benchmark case this case only introduces a time varying payout ratio. The risk-premium induces a larger average payout under the risk-neutral measure than

CASE I				
Stochastic growth - Constant volatility and risk-premia				
P def prob	Q def prob	Average Spread	Std Dev. Spread	Reg Coef
0.0144	0.043	0.0053	0.0009	3.019
(0.0005)	(0.001)			
CASE II				
Stochastic growth and volatility - Constant risk-premia				
P def prob	Q def prob	Average Spread	Std Dev. Spread	Reg Coef
0.0144	0.0501	0.0062	0.0023	2.269
(0.0006)	(0.0015)			
CASE III				
Stochastic growth and volatility - Time varying risk-premia				
P def prob	Q def prob	Average Spread	Std Dev. Spread	Reg Coef
0.0135	0.062	0.0078	0.0036	0.9
(0.0009)	(0.0018)			

Table 8: Estimated values of P and Q default probabilities as well as the unconditional mean and variance of the credit spread for four year to maturity BBB firms. Standard errors of estimates are in parenthesis. Parameters of the typical BBB firm are as defined above for the dividend claim with added 23.3% idiosyncratic volatility. The spread is simulated within a structural model which assumes a constant nominal default boundary at 60% of the average BBB leverage ratio ($K = 0.6 * 0.4328$. Upon default bond recover constant fraction of face value corresponding to average historical BBB recovery rate 51.31%. All quantities are in basis points. Simulations are run with 50000 runs for each price estimation (conditional on state), with standard antithetic variance reduction.

under the historical measure, which explains the slightly larger credit spread relative to the benchmark case, but is not sufficient to explain the data. This is in stark contrast with the equity results obtained by BY. Indeed, BY show that even with constant risk-premia long-run cash flow risk can have a sizable effect on the equity premium. This is because equity are claims to long lived cash-flows which may be subject to highly persistent shocks and thus are effectively much riskier than they might appear (i.e., in contrast to an iid consumption world) in the long run. However, corporate bonds are finitely lived. In fact, here we value relatively short term bonds, with four years to maturity. The cash flows of corporate bonds are to a large extent fixed, since only in the event of default is there effectively cash-flow risk. While the BY model is able to generate substantial variation in equity and firm value, it still remains the case that default is very infrequent and thus the cash-flow risk contributes only little to the riskiness of corporate bonds.

Case II shows that adding stochastic volatility helps predict a somewhat larger spread. Stochastic volatility simply raises the number of defaults under the risk-neutral measure. And since volatility is counter-cyclical, it shifts default probability mass from good to bad states thus keeping the average P-measure default rate identical. This helps raise spreads somewhat relative to the benchmark case. Also, stochastic volatility contributes to raising the volatility of the credit spread. However, the magnitude is not sufficient.

Instead, Case III shows the crucial role played by the time variation in risk-premia, even in the cash-flow risk model, to explain the credit spread puzzle. The BY model makes risk-premia countercyclical by linking them to volatility. This effectively makes high volatility states even 'more expensive.' Further, the model also increases the likelihood of risk-neutral defaults in high volatility states. Since this effectively shifts default events from good to bad states, the combined effects imply an increased spread for a smaller historical measure average loss rate, i.e., a lower P-measure probability of default (note that we kept the same idiosyncratic volatility to compare the three cases). The model can explain a large fraction of the spread and at 78bp is a substantial improvement over the benchmark (which predicts less than 50bp). Further, we note that the BY model because of the cash flow risk induced counter-cyclicality in defaults does not face the problem of the CC model of generating pro-cyclical P-measure default probabilities. Further, for a constant initial leverage ratio it generates a sizable standard deviation in spreads. We note that the time variation in risk-premia substantially lowers the correlation between spread and forward default probabilities. The regression coefficient of future default probabilities on spreads is 0.9 roughly in line with the empirical results (1.2).

We conclude that to explain the substantial size and time variation in credit spreads, time varying risk-aversion is essential. On the other hand, the very persistent shocks to equity values also generate substantial volatility in spreads in the BY model. On the positive side, the BY model generates the correct prediction that in bad (i.e., high volatility low growth) states the default probability is high. This is because most variation in expected default probabilities across states is due to the variation in expected cash-flows. The latter are low in bad states and high in good states, inducing counter-cyclical default probabilities in the BY model. The offsetting effect on expected returns due to the variation in risk-premia does not dominate the cash-flow effect.

We note that introducing time variation in distance to default (through variation in leverage) as investigated previously for the CC model, is not likely to improve the predictions of the BY model much. Indeed, the BY model already predicts the 'right' amount of covariation between default probabilities and credit spreads. Introducing a countercyclical default boundary will help raise spreads (by the same mechanism discussed for the CC model) but it will also substantially increase the covariation between default rates and credit spreads. So the model even with a time varying boundary cannot match both level and time series properties of spreads. It simply does not generate sufficient variation in risk-premia across states.

We conjecture however that combining CC pricing kernel with time-varying volatility will be very helpful in capturing spread level and variation.

10-year BBB and 4 and 10-year AAA spreads as well as time variation in default boundary. (to be completed).

5 Predicted time series of credit spreads

This section shows the 'out-of sample' predicted time series of credit spreads using the two models calibrated to equity data and where we extract the state variable and corresponding equity risk-premia from consumption data, risk-free rate and price-dividend ratios. As discussed earlier, we calibrate the CC model to fit moments of historical equity returns. This procedure produces a set of model parameters, and, in addition, a mapping between the surplus consumption ratio, P/D ratio, expected equity premium, and simulated BBB over AAA credit spreads. With this help, we can then back out the time series of expected equity premium and credit spread using historical data in two ways. First, we can use the historical innovation in consumption, in combination of the model parameters, to back out the time series of consumption surplus ratio, and then the corresponding expected equity premium and credit spread. Second, we can use historical P/D ratio to directly back out the time series of consumption surplus ratio, expected equity premium, and credit spreads. We then compare the simulated credit spreads with historical BBB over AAA spreads to gauge the success of our calibration. Because the consumption surplus ratio is the key state variable in CC model, we first plot in Figure 2 both this ratio backed out from consumption innovations (dubbed Ratio I) and from P/D ratio (dubbed Ratio II) for the 1919-1997 period where historical data on BBB over AAA spread is available. Ratio I closely tracks the inverse of the historical credit spread. When the credit spread reaches its peak during the great depression, Ratio I also bottoms. In comparison, Ratio II does not trace the credit spread as well (in reverse fashion). For example, Ratio II bottoms in 1950 instead of the great depression period, reflecting the fact that the historical P/D ratio is the lowest at 11.48 in 1950 and higher at 16.11 in 1933. In addition, Ratio II reaches its peak and becomes insensitive during 1959-1974 and after early 1990's because the historical P/D ratio is very high. Therefore, Figure 2 suggests that the simulated credit spread backed out from consumption innovation is more likely to track its historical counterpart than that from historical P/D ratio does.

We plot in Figure 3 the simulated equity premia and credit spreads backed out by the two methods against the true credit spread. In the upper panel of the figure, we observe that the simulated credit spread from consumption innovation indeed closely tracks its historical counterpart. They both reach the peak during the great depression, and sharply decreases afterwards and have roughly similarly variations throughout the whole period. In comparison,

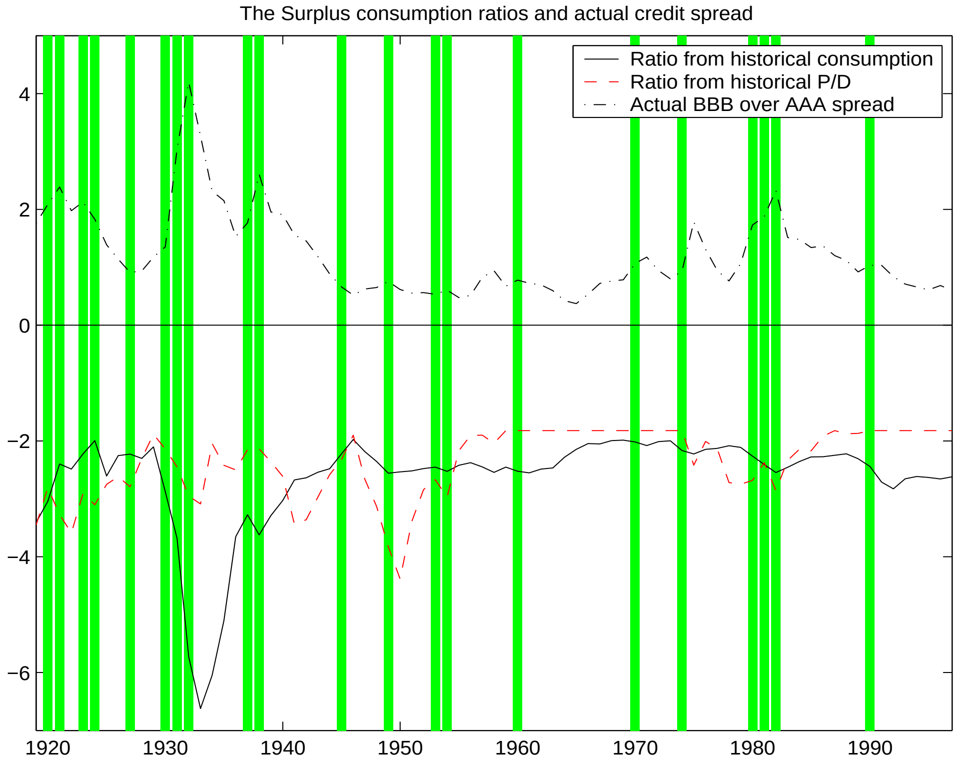

Figure 2: Time series of surplus consumption ratio of the CC model extracted using two different approaches: First, we use consumption data alone. Second we use P/D ratio.

the simulated credit spread from P/D ratio does not track the true spread as well. Similarly to the case of the consumption surplus ratio, it peaks in 1950 instead of the great depression period, and bottoms and loses its sensibility during the 1959-1974 and the 1990's period. The above observations are quantified in Table **??**, in which case we report the correlation between historical credit spread and generated surplus consumption ratio, equity premium and credit spread in the two ways. We record the correlations for both the levels and changes of all variables. Not surprisingly, these correlations are much higher for variables generated from consumption innovations than from P/D ratios. For example, the correlation between the actual and simulated credit spreads for the former (latter) case is 67% (38%) for the whole 1919-1997 period. The number reaches a striking 82% for the former and 24% for the latter case during the 1925-1976 period. However, the model seems to perform much more poorly in recent years (period 1976-1997). We find similar patterns for the correlations of the changes. The success of the consumption-innovation-generated credit spreads to capture both the level and variation of historical credit spreads is nontrivial.

This may help to shed light on how closely credit spreads are related to the equity risk premium. For example, Campbell and Taksler (2004) suggest that idiosyncratic risk can dominate the level and variation of credit spreads. On the other hand, empirical asset pricing papers often use the default spread as an empirical proxy for the equity risk premium. For example, in Jagannathan and Wang (1996), the unobservable market equity risk premium is assumed to be a linear function of Baa over AAA yield spread alone. Therefore, sorting out whether the yield spread is mainly driven by idiosyncratic or systematic risk seems important. Theoretically, the aggregate yield spread level, which is the average bond yield spread, should not reflect only systematic risk. This is because yield spread is related to total firm asset volatility. A firm with higher idiosyncratic volatility will thus, ceteris paribus, have a higher yield spread. Unlike realized equity returns, this effect will not be diversified away when a portfolio of bonds is constructed. To what extent the idiosyncratic risk components across individual firms have common trends or even factors[23] and thus credit spreads behave differently from equity risk-premia is an empirical issue.

Our results thus confirm two points. First, the historical level and variation of credit spreads can be matched in the CC model where the stochastic component is the variation in aggregate consumption, which is by definition purely driven by systematic risk. Therefore, this suggests that credit spread is likely to be mostly driven by systematic risk. Second, the close resemblance between equity premium and credit spread further suggests that historical credit spread is a good proxy for unobservable equity premium, lending some support this widely adopted practice in the literature. However, the diminished success of the model in the recent period (1976-1994) also lend some support to the anaylsis of Campbell and Taksler (2004).

6 Conclusion

In this paper we investigate whether models that are reverse engineered to fit the equity premium can explain the level and time-variation in credit spreads once they are calibrated to equity data. We compare two alternative models: the Campbell and Cochrane habit formation model which explains equity premium with time varying risk-aversion, and the Bansal and Yaron model which explains the equity premium with highly persistent shocks to expected growth and volatility of consumption. Our results suggest that highly time varying risk-premia are essential to explain the level and variation observed in spreads. However, CC's model also generates the inconsistent prediction that default probabilities are procyclical unless the default

[23]Note that even though these factors would be common, i.e., market wide, they might still not be priced in that they might be uncorrelated with aggregate consumption - in that sense idiosyncratic.

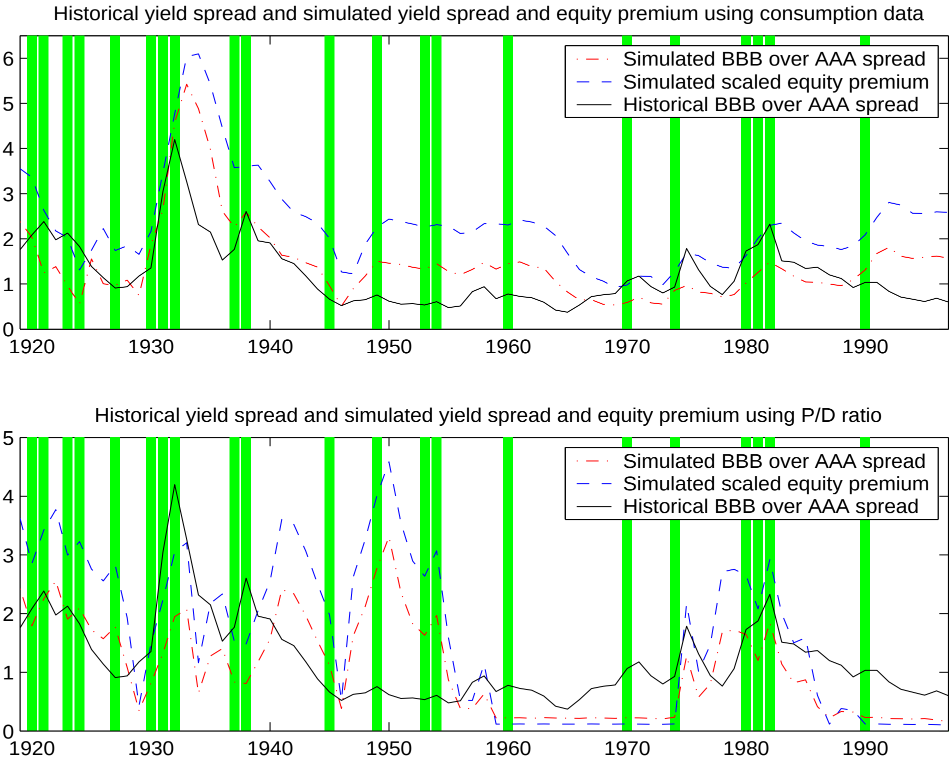

Figure 3: Time series of fitted credit spread, fitted equity premium and actual spread using two different approaches to extract the state (consumption surplus ratio) in the CC model. First, we use consumption data alone. Second we use P/D ratio.

boundary is allowed to be countercyclical. We argue that the latter is consistent with some of the existing empirical evidence (we also offer some empirical support).

Alternatively, combining insights from CC and BY may be necessary to capture adequately the behavior of spreads. Indeed, adding stochastic volatility to the CC model may help solve the pro-cyclical nature of default probabilities. Separately, it may be necessary to reconcile the predictions of the CC model with empirical evidence on the price of the consumption claim. Indeed, in the appendix, we report our estimates of the CC habit formation model, which differ substantially from the results reported in their original paper. It seems that for the CC model to reconcile historical return and volatility patterns of the post-war US economy with the aggregate consumption time series, it may require adding some shocks to growth rate and/or consumption volatility.

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A Calibration of the CC model

We report in Table 1 the simulated statistics. Columns 2 and 3 correspond to the results in CC, and the last two columns are from the historical data (postwar annual data and long term S&P 500 data respectively). A comparison of Column 2 and 8 indicates the success of the original CC study. For example, they choose parameters to match postwar annual consumption growth rate (1.88% versus 1.89%), the volatility of consumption growth (1.24% versus 1.22%), the real riskfree rate (0.94% versus 0.94%), and the Sharpe ratio (0.43 versus 0.43). The 'out-of-sample' success of their model comes from the proximity of the following three statistics that are not calibrated: the average equity premium (6.71% versus 6.69%) and the volatility of equity premium (15.64% versus 16.73%), and the price dividend ratio (18.11 versus 24.70). We argue that a better proxy for market price dividend ratio is a weighted average of equity plus corporate debt price-dividend (or coupon) ratios, which we find lowers the estimate from 24.70 to about 17. Overall, CC show that an i.i.d. consumption process, with the aid of an external equity habit, can go a long way in interpreting the equity premium puzzle.

We note that the above simulated statistics in CC are for the consumption claim. The third column in Table 1 reports the similar statistics for the dividend claim. Except for the fact that the Sharpe ratio is a bit low (0.32), other statistics are reasonably close to actual postwar data. The standard deviation of the dividend growth rate is 9.01%, close to the 11.2% of the actual data.

In Column 4 and 5, we use same parameters as in CC, but with our estimated relationship between price-dividend ratio and the state variable s. By chance, we find that $\gamma = 2$ still closely matches the observed Sharpe ratio at 0.43. More accurately, setting $\gamma = 1.95$ generates a matching Sharpe ratio. However, as noted above, the level and the standard deviation of the excess return are much lower at 3.82% and 8.89% respectively, only about half of the historical values. Furthermore, the implied price/dividend ratio is much higher (at 35.04) than historical data. Hence, we conclude that the CC model cannot capture post war security returns when calibrated to consumption dynamics.

A.1: Dividend Dynamics

We find that setting $\gamma = 6$ will yield a reasonable fit of the dividend claim, as shown in Column 7. In particular, it fits the growth rate of postwar dividend growth (2.03% versus 2.01%), the standard deviation of dividend growth (9.01% versus 11.2%), the riskfree rate (0.94% versus 0.94%), and the Sharpe ratio (0.46 versus 0.43). The level of excess equity return (5.93% versus

	Original CC		Our estimates: $\gamma = 1.95$		Our estimates: $\gamma = 6$		Historical Data	
Statistics	C-claim	D-claim	C-claim	D-claim	C-claim	D-claim	Postwar data	Long data
E[dc]	1.88	1.82	1.88	1.82	1.88	2.03	1.89	1.72
sigma(dc)	1.24	9.01	1.24	9.01	1.24	9.01	1.22	3.32
E[rf]	0.94	0.94	0.94	0.94	0.94	0.94	0.94	2.92
E(r-rf)/sigma(r-rf)	0.43	0.32	0.43	0.25	0.94	0.46	0.43	0.22
E(R-Rf)/sigma(R-Rf)	0.5	0.42	0.46	0.32	0.94	0.5	0.49	0.3
E(r-rf)	6.71	6.57	3.82	3.82	5.29	5.93	6.69	3.9
sigma(r-rf)	15.64	20.33	8.89	15.21	5.63	12.97	15.73	17.96
exp[E(p-d)]	18.11	18.51	35.04	34.53	23.03	20.72	24.7	21.16
sigma(p-d)	0.28	0.3	0.15	0.17	0.08	0.1	0.26	0.27

Table 9: Means and standard deviations of simulated and historical data.

6.69%) and its standard deviation (12.97% versus 15.73%), and the level of price dividend ratio (20.72 versus 24.70) are reasonably close to historical data. The only variable that we could not fit closely is the standard deviation of the price dividend ratio. Overall, we have shown that, once we increase the risk aversion parameter to 6, we can fit the dividend claim reasonably well. Fortunately, for our purposes, this fitting of the dividend claim is sufficient.

We find that the other properties generated in CC are still very robust. For example, in Table 2 we study whether the price/dividend ratios can still predict future equity return, another important fact recovered in many classic empirical studies. We find that, similar to CC, current price/dividend ratio or price/consumption ratio can indeed predict future returns. The predictive power increases with investment horizon. This result is important because it indicates that the primary property of the habit formation model is robust, namely, that the representative agent's risk aversion increases in the bad states, and hence demands a higher equity risk premium in order to bear such risk.

B Discrepancy with the results of Campbell and Cochrane (1999)

As noted in the text, equations (9) and (10) suggest two numerical schemes for determining the price-dividend ratio as a function of s. In particular, equation (9) can be approximated by using a recursive scheme which is iterated until the program converges to a self-consistent solution. Alternatively, Monte-Carlo methods can be used to estimate the price-dividend ratio via equation (10). Unfortunately, both methods are vulnerable to certain types of numerical

$\gamma = 2$								
Horizon (Years)	Cons. Claim 10*coef.	R^2	Div. Claim 10*coef.	R^2	Postwar data 10*coef.	R^2	Long data 10*coef.	R^2
1	-1.93	0.12	-1.98	0.09	-2.6	0.18	-1.32	0.04
2	-3.62	0.23	-3.7	0.16	-4.25	0.27	-2.77	0.08
3	-5.05	0.31	-5.18	0.21	-5.37	0.37	-3.48	0.09
5	-7.47	0.45	-7.75	0.3	-9.02	0.55	-6.04	0.18
7	-9.24	0.54	-9.67	0.35	-12.11	0.65	-7.54	0.23
10	-11.41	0.64	-12.14	0.41	-16.37	0.8	-9.25	0.24

$\gamma = 6$								
Horizon (Years)	Cons. Claim 10*coef.	R^2	Div. Claim 10*coef.	R^2	Postwar data 10*coef.	R^2	Long data 10*coef.	R^2
1	-1.55	0.07	-1.65	0.03	-2.6	0.18	-1.32	0.04
2	-2.9	0.13	-3.06	0.06	-4.25	0.27	-2.77	0.08
3	-4.01	0.17	-4.25	0.08	-5.37	0.37	-3.48	0.09
5	-5.93	0.26	-6.44	0.11	-9.02	0.55	-6.04	0.18
7	-7.31	0.32	-8.14	0.13	-12.11	0.65	-7.54	0.23
10	-9.02	0.39	-10.42	0.16	-16.37	0.8	-9.25	0.24

Table 10: Long horizon return regressions for $\gamma = 2$ and $\gamma = 6$

error, as we now demonstrate.

Recursive schemes are vulnerable to numerical errors because in order to implement them, one must discretize the state variable s and then specify some set of lower and upper bound cut-offs. While the discretization process can potentially introduce errors, these errors can usually be controlled by obtaining estimates for different sizes of increments, and then extrapolating back to the limit $ds \to 0$. In contrast, specifying finite values for the cut-off values $\underline{s}$ and $\overline{s}$ can lead to errors that are more difficult to control. Typically the best that one can do here is to see whether the estimated solution is converging as one increases the range $s \in (\underline{s}, \overline{s})$. We note that CC use the recursive method to estimate the price-dividend ratio. Using their own code, we demonstrate below that the estimated solution changes dramatically as $\underline{s}$ is lowered.

Monte Carlo methods are also vulnerable to numerical error, as can be seen from this illustrative example. Assume a random variable $\tilde{X}$ can take on only two values: 10^{12} with the probability $p\left(\tilde{X} = 10^{12}\right) = 10^{-12}$, and zero. Clearly, the true expected value is one, and the variance is approximately 10^{12}. However, if one simulates only, say, 10^6 paths, then it is very likely that the simulation will generate a sample mean and sample variance of zero! This example emphasizes that the implied standard error might be a poor indicator of whether or not a sufficient number of sample paths have been run. It is worth noting that Monte Carlo approaches do not bias the mean estimate, but they may bias the estimate for the 'typical' or 'median' path. As an example more relevant to the current situation, if a given random variable is normal $\tilde{X} \sim (0,1)$, and one attempts to estimate $\mathrm{E}\left[e^{aX}\right]$, the number of sample paths necessary for convergence increases significantly with a, even though the implied standard error may not give an indication that convergence has not been obtained. As we demonstrate below, it is the 'long-tail' of the pricing kernel Λ which makes the Monte Carlo approach fail here, at least for reasonable numbers of sample paths. Indeed, the price of a security can be calculated as the expectation of the product of the pricing kernel and the state-dependent cash flows.

$$\begin{aligned} P &= \mathrm{E}\left[M\,X\right] \\ &= \int_{-\infty}^{s_{max}} ds_T\, \pi\left(s_T \,\middle|\, s_t \right) M(s_T)\, X(s_T) \\ &= \int_{-\infty}^{s_{max}} ds_T\, \pi\left(s_T \,\middle|\, s_t \right) e^{-\alpha(T-t)} e^{-\gamma\left[(s_T+C_T)-(s_t+c_t)\right]}\, X(s_T). \end{aligned} \tag{B.66}$$

We claim that, due to the long tail, finite sample Monte Carlo estimation methods bias downward those probabilities $\pi\left(s_T \,\middle|\, s_t \right)$ for large negative s_T, where marginal utility is higher, and bias upward those probabilities $\pi\left(s_T \,\middle|\, s_t \right)$ for less negative values of s_T, where marginal utility is lower, leading to a downward biased estimate for the price (or equivalently, the price-dividend ratio) of a security for the 'typical' Monte Carlo simulation.

Thus, the problem that the Monte Carlo approach runs into can be traced back to the long tail of the distribution of s. However, for this particular model, there is a simple way to circumvent this difficulty. Indeed, here we demonstrate that by transforming from the 'historical measure' to the 'risk-neutral measure', the problems associated with the 'long-tail' of the pricing kernel are eliminated. Indeed, we find that the Monte Carlo approach under the Q-measure generates a solution which is very well behaved, even for a relatively low number of sample paths. Specifically, we can re-write equations (9)-(10) as

$$\left(\frac{P(t)}{C(t)}\right) = \mathrm{E}_t^Q\left[\left(\frac{1}{R_{t,t+1}}\right)\frac{C(t+1)}{C(t)}\left(1+\frac{P(t+1)}{C(t+1)}\right)\right] \tag{B.67}$$

$$= \sum_{j=1}^{\infty} \mathrm{E}_t^Q\left[\left(\frac{1}{\prod_{m=0}^{j-1} R_{(t+m),\,(t+m+1)}}\right)\frac{C(t+j)}{C(t)}\right]. \tag{B.68}$$

The reason why this transformation is useful is that, even though the pricing kernel has a long tail, its expectation (for values of $s < s_{max}$) generates a one-period risk-free rate that is constant.

We demonstrate the advantage of this transformation by estimating the price-consumption

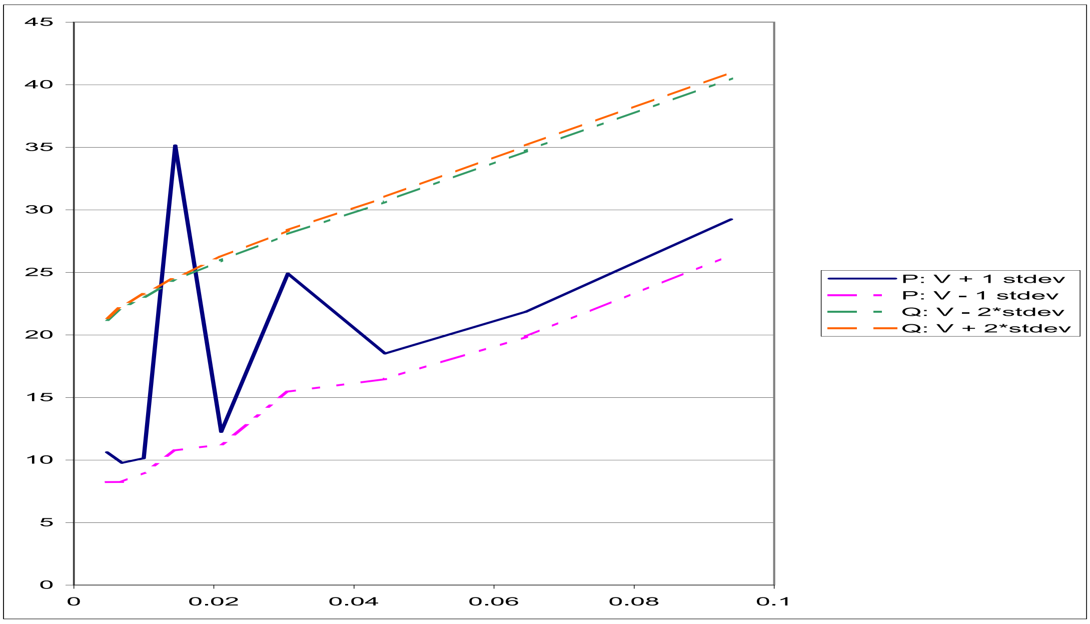

Figure 4: Estimation of the price-consumption ratio using Monte Carlo methods under both the P and Q-measures. Parameter values are $g = 0.0189$, $\gamma = 2$, $\sigma = 0.015$, $\kappa = 0.138457$, and $r = 0.0094$. 100,000 sample paths are used for the P-measure estimates, whereas only 10,000 sample paths are used for the Q-measure estimates.

ratio using Monte Carlo methods under both the P-measure, and the Q-measure. As demonstrated in Figure (4), we see that the Q-measure Monte Carlo estimation using only 10,000 sample paths generates a smooth, monotonic function whose standard error is so low that we need to add plus-or-minus two standard errors in order for the two curves to be distinguishable. In contrast, even after 100,000 sample paths under the P-measure, the estimates are still very noisy and non-monotonic. Interestingly, we note that the P-measure estimates are rather similar to those obtained by CC, whereas the Q-measure estimates generate significantly different values, especially for low values of s. Below, we argue that the Q-measure estimates are in fact the true price-consumption ratio in the CC economy.

One advantage of the Monte Carlo approach is that one may price each dividend claim separately. In Figure (5), we plot the P- and Q-measure estimates for the prices of the individual consumption claims as a function of maturity. We find that the Q-measure estimate generates

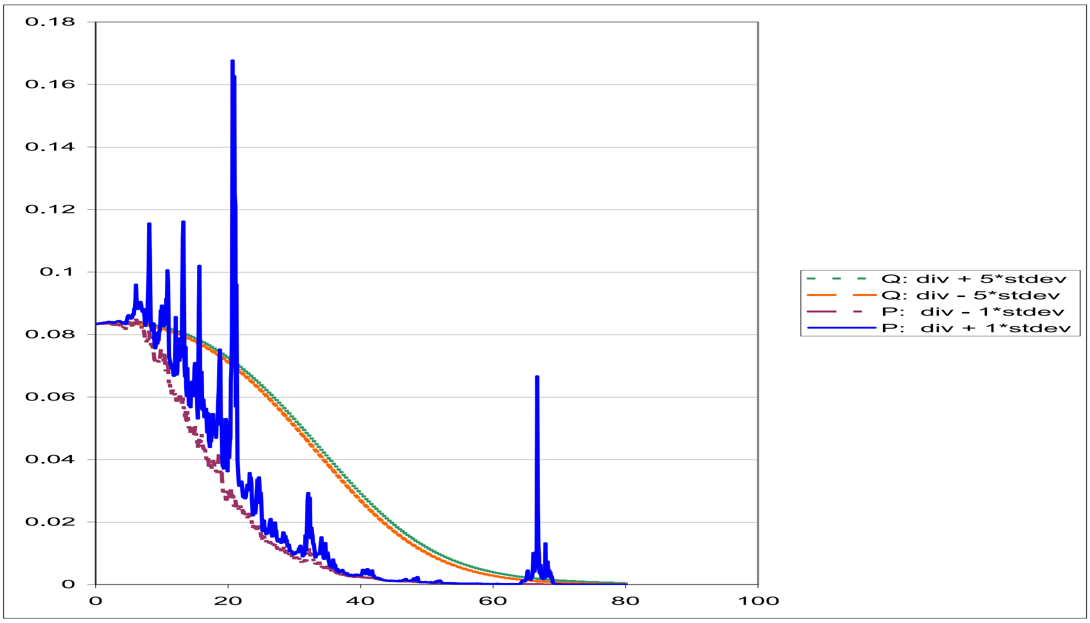

Figure 5: Estimation of the value of the individual consumption claims using Monte Carlo methods under both the P and Q-measures. Parameter values are $g = 0.0189$, $\gamma = 2$, $\sigma = 0.015$, $\kappa = 0.138457$, and $r = 0.0094$. 100,000 sample paths are used for the P-measure estimates, whereas only 10,000 sample paths are used for the Q-measure estimates.

a very smooth function. Indeed, we needed to add plus-or-minus five standard errors to create two lines that were distinguishable. In contrast, the P-measure estimates generate a rather noisy function, even though we used ten-times more sample paths to estimate it.

Of course, smoothness alone does not guarantee that our Q-measure estimates are accurate. In order to provide more convincing evidence that the Q-measure estimates are the solution to the CC model, here we examine a one-period model in a CC framework to price both the claim to \$1, that is, a risk-free bond, and a claim to next period's consumption $C(t+1)$. The advantage of a one period model is that, since the CC model is specified to have a log-normal distribution, the true solution is known in analytical form.

In Figure 6 we plot P-estimates (plus or minus one standard error) for the value of the one period consumption claim, along with the actual value. We do not plot the Q-value estimates, because even plus or minus 10 standard errors would generate a curve indistinguishable from

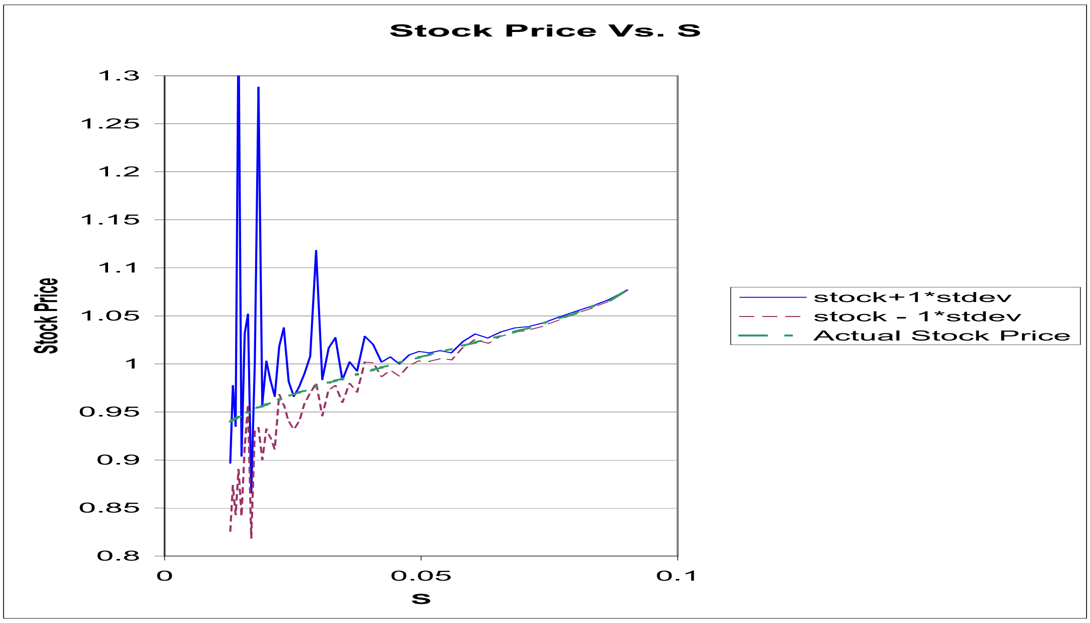

Figure 6: P-estimation for the value of the claim to one period's consumption. The three curves are the true value, and the estimated value plus or minus one standard error. Parameter values are $g = 0.0189$, $\gamma = 2$, $\sigma = 0.015$, $\kappa = 0.138457$, $r = 0.0094$ and $dt = 10$. 100,000 sample paths are used for the P-measure estimates.

the actual answer. Admittedly, both the P- and Q-estimates give excellent answers when the time increment is on the order of one month. However, in order to capture the point that pricing a security implies pricing accurately the dividend claim for many decades, we chose $dt = 10$ years for this figure. We emphasize, however, that the one period model is exact regardless of the time increment. This finding strongly suggests that the Q-measure Monte Carlo estimates given above in Figure 4 are provide excellent numerical estimates of the actual value.

Analogously, we priced the one-period bond as a function of s under the P-measure. Note that under the Q-measure, the solution is exact by construction. Once again, we see that for large negative values of s the Monte Carlo estimates are poor.

Finally, in Figure 8 we plot the estimated value of the price-consumption ratio using the computer program of CC, available on their web page. In the figure we plot the price/dividend

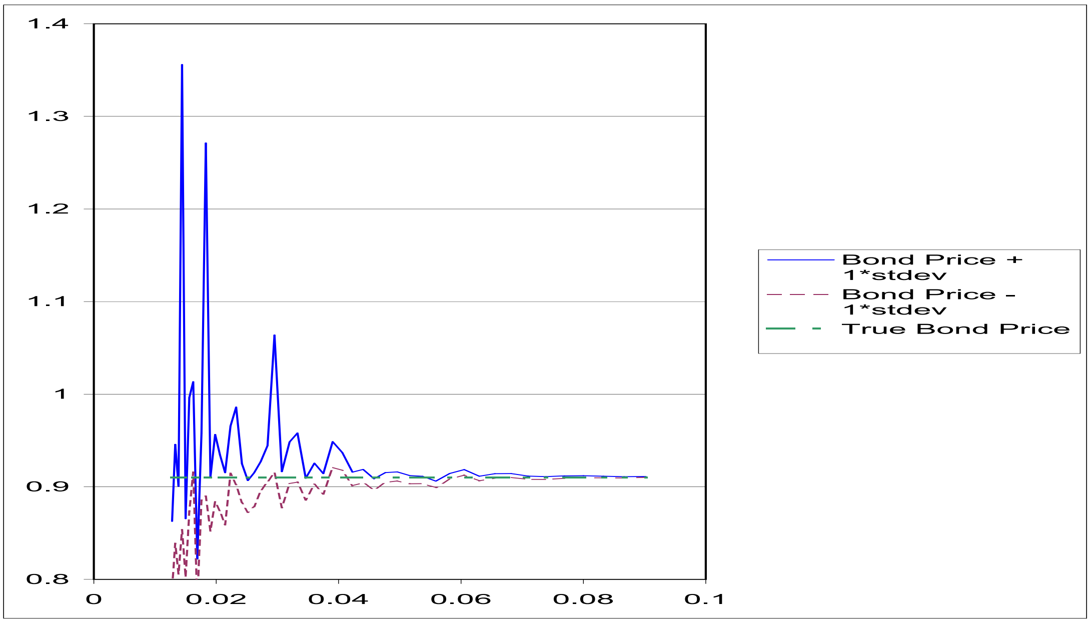

Figure 7: P-estimation for the value of the one-period riskless bond. The three curves are the true value, and the estimated value plus or minus one standard error. Parameter values are $g = 0.0189$, $\gamma = 2$, $\sigma = 0.015$, $\kappa = 0.138457$, $r = 0.0094$ and $dt = 10$. 100,000 sample paths are used for the P-measure estimates.

ratio and price/consumption ratio using both the cutoff points as in CC and lower cutoff points. It is clear that, in the original version of CC, the level of the ratios are much lower than what we obtain through the Monte Carlo simulation with the risk-neutral pricing formula. On the other hand, when much lower cutoff points are used, the ratios are much higher and very similar to those obtained for the risk-neutral Monte Carlo estimates. This comparison indicates that, when the proper cutoff points are used, the two methods will yield similar results.

C A continuous time solution to the CC model

to be completed

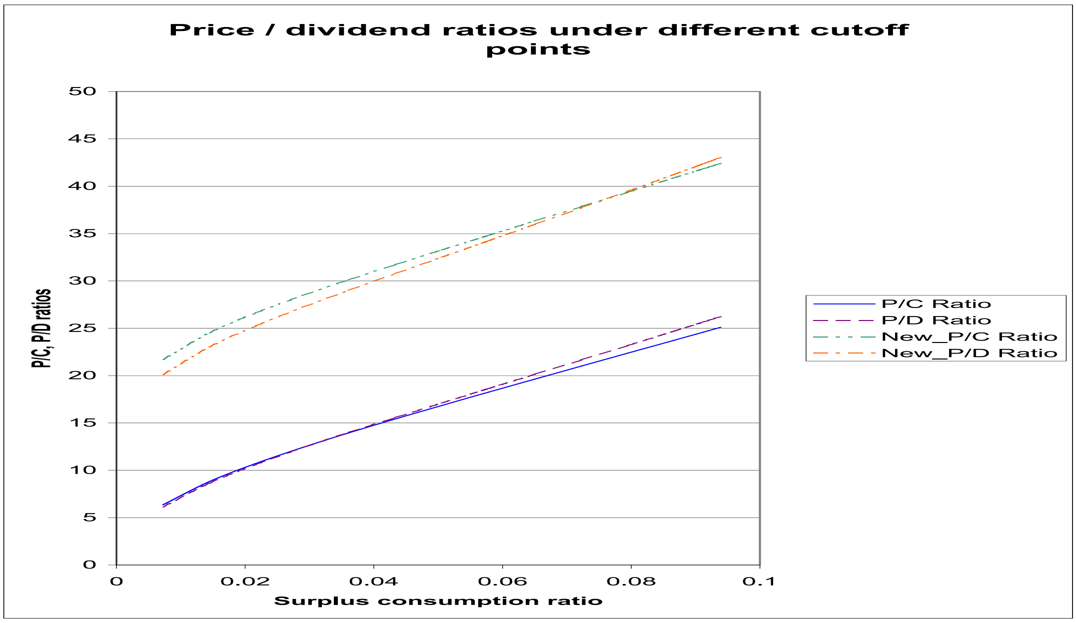

Figure 8: P and Q-estimations using the iterative procedure of CC. Two different values are used for $\underline{s}$: -5 and -50.

D A variance reduction technique for first passage time simulations based on 'importance sampling'

to be completed