

# Correlated Defaults and the Valuation of Defaultable Securities

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## Motivation

- Modeling correlated defaults is a top priority in credit risk research.
  - The valuation of certain credit derivatives, such as basket credit default swaps, is sensitive to default correlation in the underlying portfolio.
  - Banks need to set aside capital against defaults in their loan portfolio by computing the credit VaR.
  - The pricing of complex structures may require consideration of counterparty risk.

## Current methods

- Copula-based approaches:
  - Marginal distribution can be from any model, and default dependency summarized by copula functions.
  - Li (2000), Schonbucher and Schubert (2001).
- Structural approaches:
  - Correlation of asset levels and default thresholds.
  - Hull and White (2001), Giesecke (2001).
- Intensity-based approaches:
  - Common factors in default intensities, and firm-specific counterparty risk.
  - Partial solution offered by Jarrow and Yu (2001) for a simplified default dependency structure.

## Our contribution

- We fully develop the intensity-based approach.
  - A recursive procedure can be used to simulate default times with intensities dependent on common factors and past defaults.
- We compare this approach to existing methods.
  - It can be considered as an extension of Schonbucher and Schubert (2001) where the copula function is allowed to vary with the sample path of the common factors.
- We apply this approach to the valuation of
  - defaultable bonds,
  - credit default swaps,
  - and basket credit default swaps.

## Preliminaries

- A typical reduced-form model assumes that the default time  $\tau$  is defined as

$$\tau = \inf \left\{ t : \int_0^t \lambda_s ds \geq E \right\},$$

where  $\lambda$  is the default intensity dependent on exogenous common factors  $X$ , and  $E$  is a unit exponential random variable independent of  $X$ .

- $N_t = 1_{\{t \geq \tau\}}$  defines a Cox process representing default, with intensity  $\lambda$ .

## The “conditionally independent construction”

- With multiple default times  $(\tau^1, \dots, \tau^I)$ , this definition can be extended with a collection of iid unit exponential random variables  $(E^1, \dots, E^I)$ .
- This implies that the default times are independent conditional on the information contained in  $X$ , and default correlation arises because of correlation of the default intensities.
- Hull and White (2001) and Schonbucher and Schubert (2001) have expressed doubt on the ability of this type of models to match the level of empirical default correlation.

## Basic setup

- We consider  $\tau = (\tau^1, \dots, \tau^I)$  defined on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  where  $\mathcal{F}_t$  is the minimal sigma-field needed to support both  $X$  and  $\tau$ .
- We assume that  $\tau^i$  has  $(P, \mathcal{F}_t)$ -intensity  $\lambda^i$ .
- In other words,  $\lambda^i$  is the conditional hazard rate of  $\tau^i$ :

$$\begin{aligned} \lambda_t^i 1_{\{t < \tau^i\}} &= \lim_{\Delta t \rightarrow 0^+} \frac{E(N_{t+\Delta t}^i - N_t^i | \mathcal{F}_t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0^+} \frac{P(t < \tau^i \leq t + \Delta t | \mathcal{F}_t)}{\Delta t}. \end{aligned}$$

## Some technical issues

- Unlike the “conditionally independent construction,”  $N_t^i = 1_{\{t \geq \tau^i\}}$  is no longer a Cox process.

- Generally (Kusuoka 1999), the survival probability

$$P(\tau^i > T | \mathcal{F}_t) \neq 1_{\{\tau^i > t\}} E \left( \exp \left( - \int_t^T \lambda_s^i ds \right) | \mathcal{F}_t \right).$$

- The conditional expectation above jumps at  $\tau^i$ , violating the “no-jump condition” of Duffie, Schroder and Skiadas (1996).
- Although a “trick” (CGH 2003) can be used to restore this formula under a change of measure, anything that requires the joint distribution of default times can pose a problem.

## How to construct the default times?

- We use the “total hazard construction” of Norros (1986) and Shaked and Shanthikumar (1987).
- Given their conditional hazard rates, this method generates stopping times from iid unit exponentials.
- It results from the fact that the conditional hazard rates integrated to the stopping times are iid unit exponential random variables.
- We extend this construction so that the conditional hazard rates can also depend on exogenous common factors, a case of broader appeal.

## Case 1: “Internal histories”

- Let  $\lambda_i(t|n)$  denote the intensity for  $i$  given  $n$  past defaults,  $t_{k_1}, \dots, t_{k_n}$ , where  $0 = t_{k_0} < t_{k_1} < \dots < t_{k_n} < t < \tau^i$ .
- Define the total hazard accumulated by  $i$  by time  $t$  given  $n$  observed defaults as

$$\psi_i(t|n) = \sum_{m=1}^n \Lambda_i(t_{k_m} - t_{k_{m-1}} | m-1) + \Lambda_i(t - t_{k_n} | n),$$

where  $\Lambda_i(s|m) = \int_{t_{k_m}}^{t_{k_m}+s} \lambda_i(u|m) du$  is the total hazard accumulated by  $i$  for a period of length  $s$  following the  $m$ th default.

- Also define an inverse function:

$$\Lambda_i^{-1}(x|n) = \inf \{s : \Lambda_i(s|n) \geq x\}, \quad x \geq 0.$$

The following recursive procedure constructs a collection of random variables  $\widehat{\tau} = (\widehat{\tau}^1, \dots, \widehat{\tau}^I)$  with the same joint distribution as the original default times.

1. Draw a collection of iid unit exponentials  $E = (E^1, \dots, E^I)$ .

2. Let

$$k_1 = \arg \min_{1 \leq i \leq I} \left\{ \Lambda_i^{-1} (E^i | 0) \right\},$$

$$\widehat{\tau}^{k_1} = \Lambda_{k_1}^{-1} (E^{k_1} | 0).$$

3. Assume that the values of  $(\widehat{\tau}^{k_1}, \dots, \widehat{\tau}^{k_{m-1}})$  are already given, where  $m \geq 2$ . Define the set  $I_{m-1} = \{k_1, \dots, k_{m-1}\}$  and  $\bar{I}_{m-1}$  as the set of firms excluding  $I_{m-1}$ . Recall that  $\psi_i(t | m-1)$  is the total hazard accumulated by firm  $i$  given the first  $m-1$  defaults, let

$$k_m = \arg \min_{i \in \bar{I}_{m-1}} \left\{ \Lambda_i^{-1} \left( E^i - \psi_i(\widehat{\tau}^{k_{m-1}} | m-1) | m-1 \right) \right\},$$

$$\widehat{\tau}^{k_m} = \widehat{\tau}^{k_{m-1}} + \Lambda_{k_m}^{-1} \left( E^{k_m} - \psi_{k_m}(\widehat{\tau}^{k_{m-1}} | m-1) | m-1 \right).$$

4. If  $m = I$  then stop. Otherwise, increase  $m$  by 1 and go to Step 3.

## Case 2: External common factors

- Denote by  $\omega \in \mathcal{F}_\infty^X$  a complete sample path of the exogenous common factors  $X$ .
- Given  $\omega$ , the intensities collapse into functions with explicit dependence on the default times.
- The previous procedure can be used to construct random variables  $\hat{\tau}^i$  with  $\mathcal{G}_t$ -conditional hazard rate  $\lambda^i$ , where  $\mathcal{G}_t = \mathcal{F}_\infty^X \vee \mathcal{F}_t^1 \vee \dots \vee \mathcal{F}_t^I$ .
- Since we assume  $\lambda^i$  to be  $\mathcal{F}_t$ -predictable,  $\hat{\tau}^i$  has  $\mathcal{F}_t$ -conditional hazard rate  $\lambda^i$  as well.
- Replace Step 1 with “Draw a complete sample path of  $X$ . Draw iid unit exponentials  $E = (E^1, \dots, E^I)$  independent of  $X$ .”

## A specific copula-based approach

- Schonbucher and Schubert (2001) define  $\tau^i$  as

$$\tau^i = \inf \left\{ t : \int_0^t h_s^i ds \geq E^i \right\},$$

where  $h^i$  is an  $\mathcal{F}_t^X$ -adapted process.

- They assume that the unit exponentials  $(E^1, \dots, E^I)$  are governed by a joint distribution function  $C(U^1, \dots, U^I)$ , where  $U^i = \exp(-E^i)$  and  $C$  is a copula.
- The “conditionally independent construction” corresponds to choosing the product copula:

$$C(U^1, \dots, U^I) = U^1 U^2 \dots U^I.$$

## Relationship between the two approaches

- If only the internal history of default is observed, the two approaches are equivalent in the following sense:
  - Given the joint distribution  $F$ , we can always find the marginal distributions and a copula  $C$  that stitches the marginals into  $F$ .
  - Given the joint distribution  $F$ , we can always find the conditional hazard rates and apply the total hazard construction to recover  $F$ .
- If we also observe  $X$ , then the joint distribution and the associated copula are conditional on  $\omega \in \mathcal{F}_\infty^X$ .
  - In this case, our approach is an extension of Schonbucher and Schubert (2001).

## Example 1: Defaultable bond pricing, $I = 2$

- Two default times  $\tau^A$  and  $\tau^B$ , with the following intensities:

$$\lambda_t^A = a_1 + a_2 1_{\{t \geq \tau^B\}},$$

$$\lambda_t^B = b_1 + b_2 1_{\{t \geq \tau^A\}}.$$

- Jarrow and Yu (2001) solve a simplified case:

$$\lambda_t^A = a,$$

$$\lambda_t^B = b_1 + b_2 1_{\{t \geq \tau^A\}}.$$

- Complete solution can be derived by total hazard construction (alternative solutions given by Kusuoka 1999 and CGH 2003).
- Pricing of  $B$  coincides with the simplified case of JY (Bielecki and Rutkowski 2002).
  - We can ignore the default dependency of  $A$  when pricing  $B$ .

## Example 2: Defaultable bond pricing, $I = 3$

- Three default times:

$$\lambda_t^A = a_{10} + a_{12}1_{\{t \geq \tau^B\}} + a_{13}1_{\{t \geq \tau^C\}},$$

$$\lambda_t^B = a_{20} + a_{21}1_{\{t \geq \tau^A\}} + a_{23}1_{\{t \geq \tau^C\}},$$

$$\lambda_t^C = a_{30} + a_{31}1_{\{t \geq \tau^A\}} + a_{32}1_{\{t \geq \tau^B\}}.$$

- Can be applied to the pricing of CDS, where the buyer, the seller, and the underlying all have credit quality issues.
- When pricing  $A$ , we can ignore the impact of  $A$  on  $B$  and  $C$ , but we should not ignore the default dependency between  $B$  and  $C$ .

- An alternative solution is to use the CGH formula:

$$P(\tau^A > t) = E' \left( \exp \left( - \int_0^t \lambda_s^A ds \right) \right)$$

under a change of measure to  $P'$  that puts zero probability on  $\{\tau^A \leq t\}$ .

- The expectation above should be evaluated by assuming a joint distribution for  $\tau^B$  and  $\tau^C$  determined by the following conditional hazard rates:

$$\begin{aligned} \lambda_t^B &= a_{20} + a_{23} 1_{\{t \geq \tau^C\}}, \\ \lambda_t^C &= a_{30} + a_{32} 1_{\{t \geq \tau^B\}}. \end{aligned}$$

Note that  $\tau^A$  has been dropped from this system.

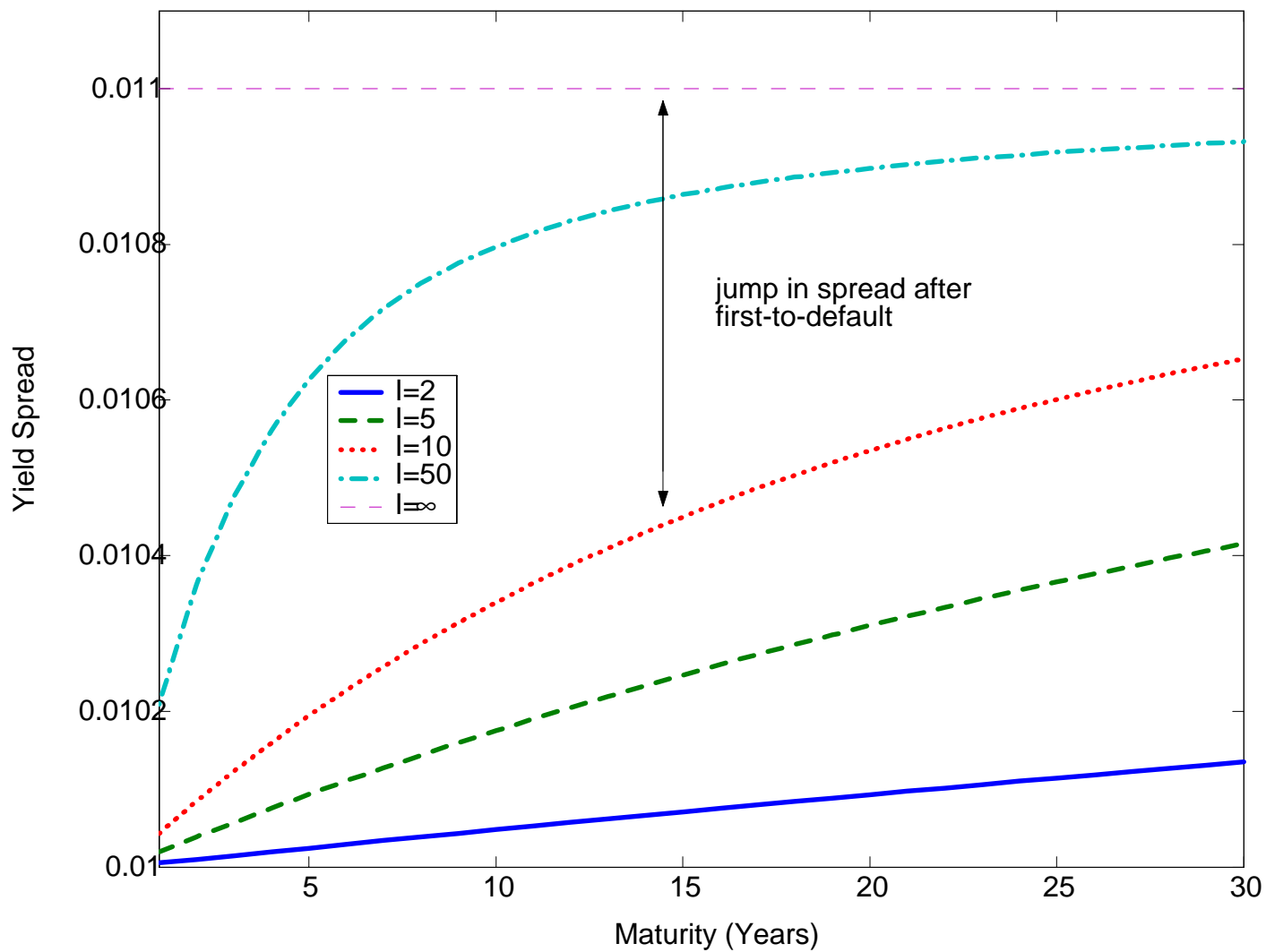
### Example 3: Defaultable bond pricing, FTD

- Empirical evidence shows market-wide response to major credit events.
  - Lang and Stulz (1992): bankruptcy filings.
  - Newman and Rierson (2003): large debt offerings.
  - Collin-Dufresne, Goldstein and Helwege (2003): large change in credit spreads.
- We consider the following intensities:

$$\lambda_t^i = a_1 + a_2 1_{\{t \geq \tau_F\}},$$

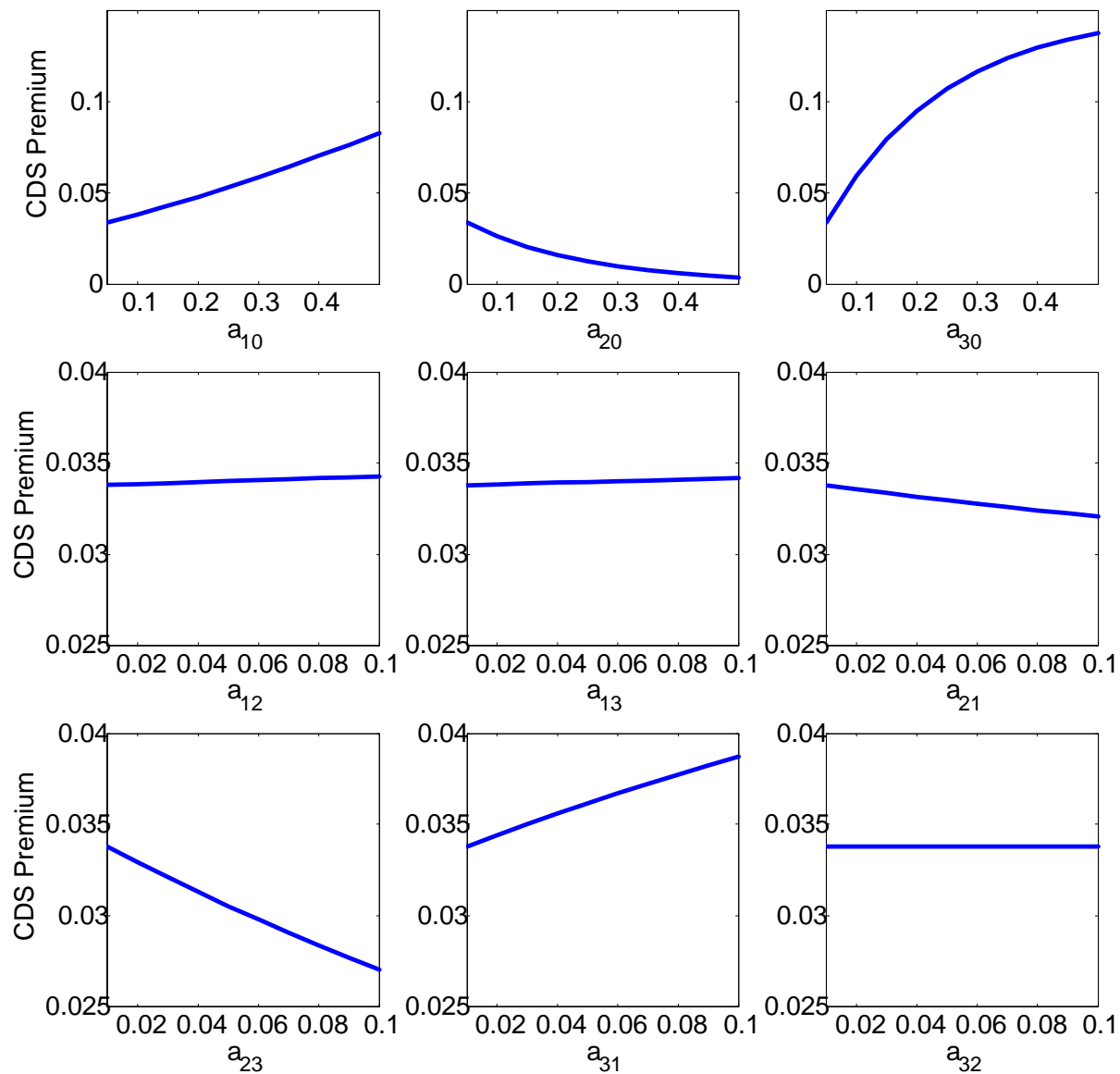
where  $\tau_F = \min(\tau^1, \dots, \tau^I)$  is the first-to-default time.

- This model can be calibrated to bond price changes around credit events.



## Example 4: CDS pricing with counterparty risk

- In contrast to the typical arrangement, we assume that the buyer will pay until its own default or the expiration of the CDS, whichever is earlier.
- We also assume that the default protection is paid at CDS expiration.
- Generally, anything that elevates the credit risk of the buyer or reference asset, or reduces the credit risk of the seller, will increase the CDS premium.
  - This can be a direct effect or an indirect effect.

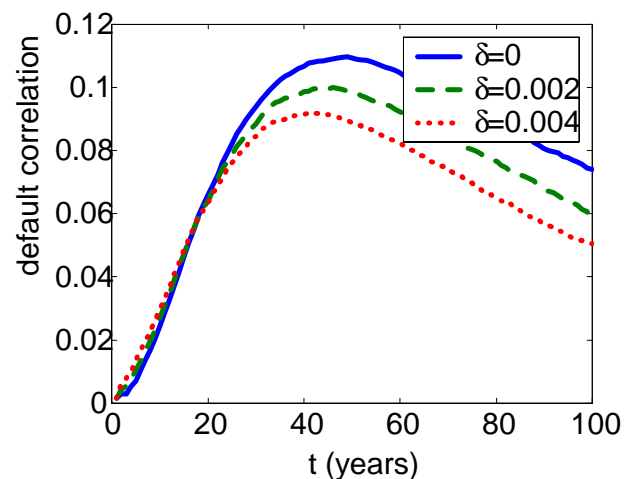
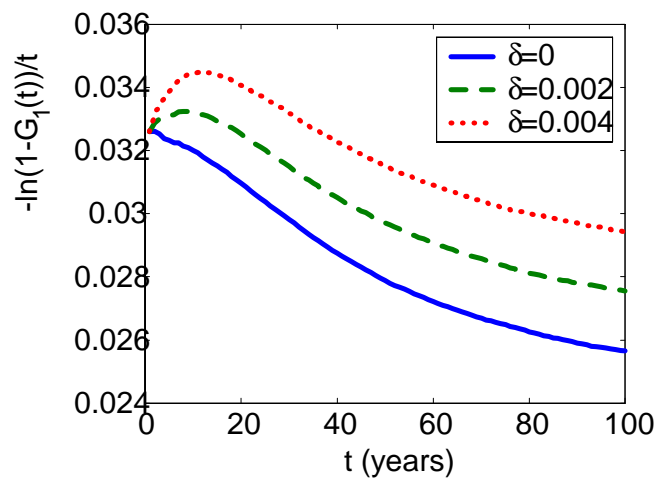
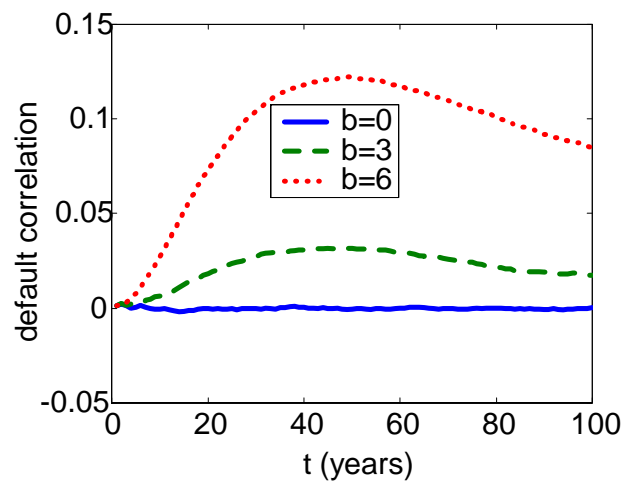
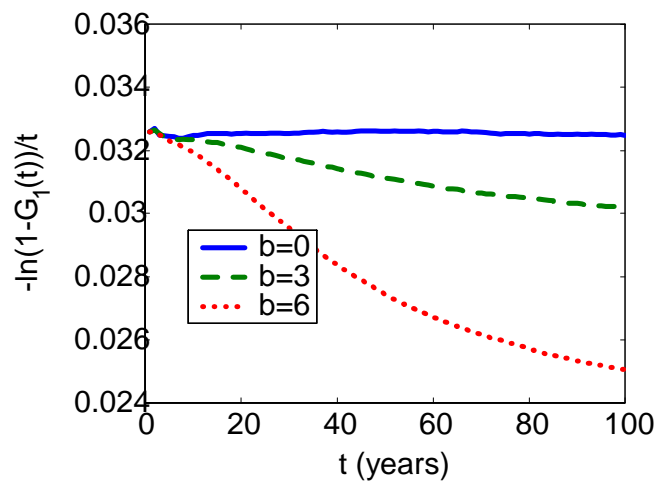


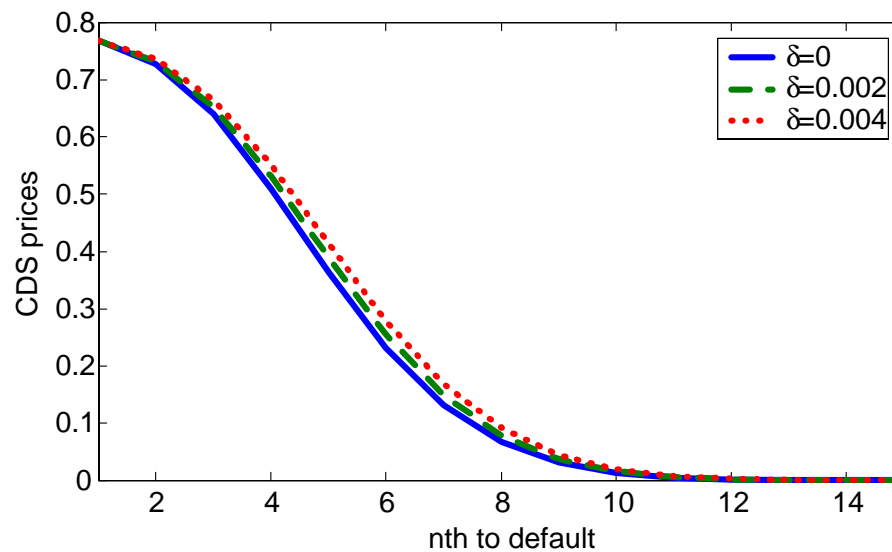
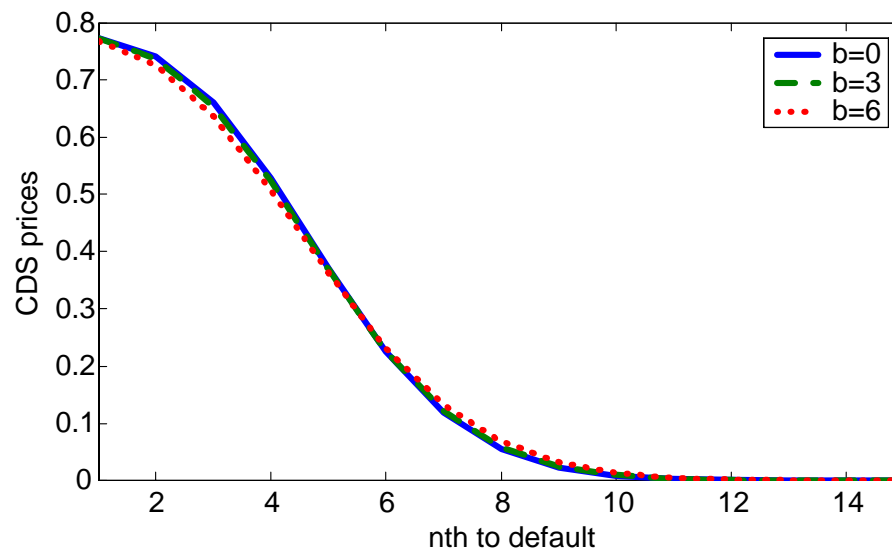
## Example 5: Pricing basket CDS

- Basket CDS pays \$1 at expiration if the  $n$ th-to-default in a portfolio occurs prior to expiration.
- The premium is proportional to  $P(\tau^{(n)} \leq T)$ .
- We assume the following state-dependent intensity for  $I$  reference credits:

$$\lambda_t^i = a + bF_t + \delta 1_{\{t \geq \tau^{(1)}\}}.$$

- Both the common factor and the default contagion can contribute to default correlation in this portfolio.





## Conclusion

- We introduce an intensity-based model of correlated defaults that offers more flexibility than models based on Cox processes.
- A simulation-based approach can be used to generate default times with intensities that depend on external common factors and past defaults.
- This approach is equivalent to a copula-based approach where the copula function is allowed to be “dynamic,” changing with the histories of exogenous state variables.
- Further research is needed to fully understand the correspondence between intensity-based and copula-based models.