Perspectives on Pricing in the Presence of Default

by

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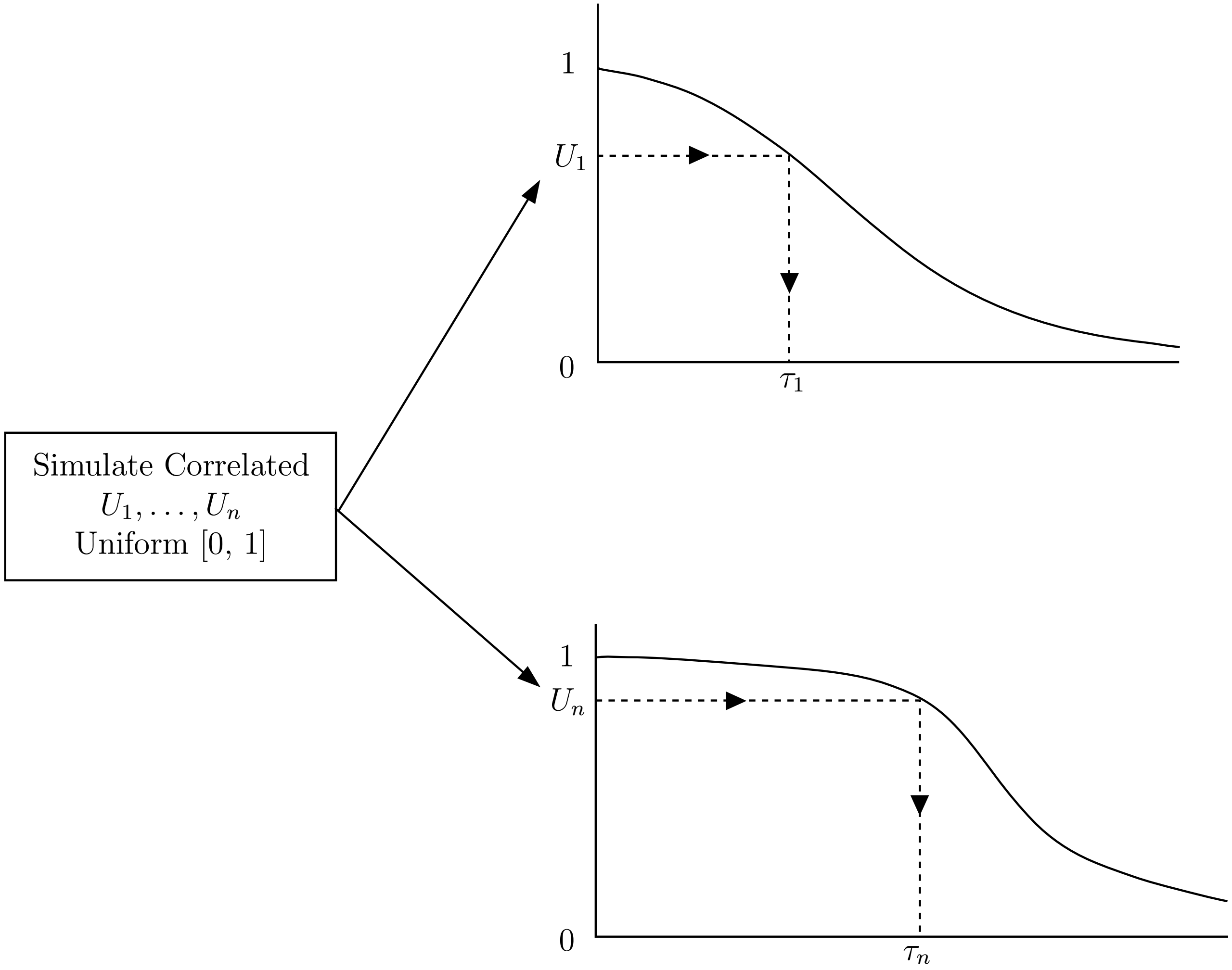
- Correlated Default Times in Marked-to-Market Portfolio Analysis and Basket Pricing
- The Market Price of Default Risk
- Recovery

Correlated Default and Marked-to-Market

- Let $V_i(t)$ denote the market value of loan i at time t.
- We will want to calculate $P(V_1(t) + \cdots + V_n(t) \leq k)$, or the price of an option on $V_1(t) + \cdots + V_n(t)$.
- E.g., the values of Trax and Iboxx positions depend not only on which counter-parties have defaulted, but also on the spreads of those issuers that have survived.
- The joint revaluation risk should include both correlated default and correlated uncertain changes in spreads.



Simulate Correlated
$U_1, \ldots, U_n$
Uniform [0, 1]

Figure 1: Copula-based simulation of correlated default times

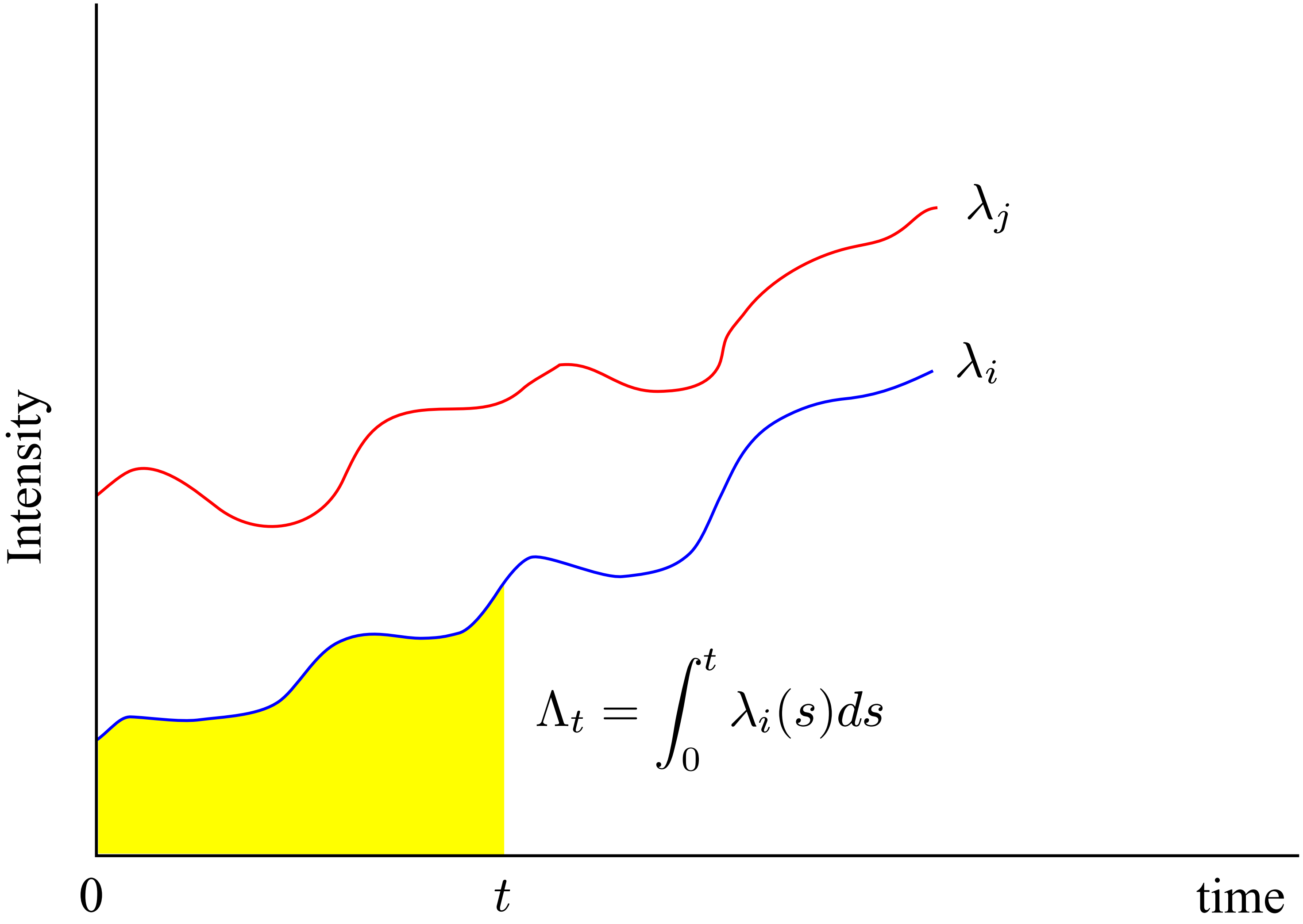


Figure 2: Intensity-based simulation of correlated default times

Do Intensity-Based Model Generate Enough Default Event Correlation?

- Intensity-based models allow for state-dependent changes in the spreads of the underlying issuers– spreads are directly related to the risk-neutral intensities.
- $Corr(\lambda_i(t), \lambda_j(t)) > 0 \Rightarrow$ implies positive default time correlations. Will this be "enough" to price structured products?
- Maybe, depending on factor structure of intensities?
- Maybe, if $Vol^{\mathbb{Q}}(\lambda^{\mathbb{Q}}) > Vol^{\mathbb{P}}(\lambda^{\mathbb{Q}})$?

Discrete-Time Affine Pricing Models

- Qiang Dai (NYU), Anh Le (NYU), and I are developing discrete-time "affine" models of the risk factors X:

  $$f^{\mathbb{P}}(X_{t+1}|X_t) = f^{\mathbb{Q}}(X_{t+1}|X_t) \times (d\mathbb{P}/d\mathbb{Q})_{t,t+1}.$$

  where $(d\mathbb{P}/d\mathbb{Q})_{t,t+1}$ is the *Radon-Nikodym* derivative.

- A specification of $(d\mathbb{P}/d\mathbb{Q})_{t,t+1}$ that *nests* extant continuous-time formulations is

  $$(d\mathbb{P}/d\mathbb{Q})_{t,t+1} = \frac{e^{\Lambda_t' X_{t+1}}}{\phi^{\mathbb{Q}}(\Lambda_t; X_t)}, \text{ where}$$

  $\Lambda_t = \Lambda(X_t)$ is the *market price of risk.*

- $f^{\mathbb{P}}$ is also known in closed-form– facilitates calibration.
- Accommodates stochastic volatility, switching regimes, etc.

Is Default Risk Priced?

- There are subtle issues involved in answering this question, because we must use both price data (informative about $\lambda^{\mathbb{Q}}$) and historical default information (informative about $\lambda^{\mathbb{P}}$).
- These two default intensities are related by

$$\lambda_t^{\mathbb{Q}} = (1 - \Gamma_t)\lambda_t^{\mathbb{P}}.$$

- Importantly, the requirement of no arbitrage places only weak restrictions on the risk premium Γ_t. One intensity might jump and the other not, one may have stochastic volatility and the other not have it, etc.

- The instantaneous excess return on a defaultable zero-coupon is with price $B(t,T)$ is:

$$e_{Bt} = \frac{1}{B(t,T)} \frac{\partial B(t,T)}{\partial X'} \sigma_X \Lambda_t + \frac{w_t - B(t,T)}{B(t,T)} \lambda_t^{\mathbb{P}} \Gamma_t.$$

- The first component captures the risk associated with the state variables driving r_t, $\lambda^{\mathbb{Q}}$, and $L^{\mathbb{Q}}$.
- Note that the risk of a changing likelihood of default is captured in this term! How does $\lambda^{\mathbb{Q}} L^{\mathbb{Q}}$ change with X?
- The term $\frac{w_t - B(t,T)}{B(t,T)} \lambda_t^{\mathbb{P}} \Gamma_t$ represents compensation for the expected loss due to actual default:
  - $\frac{w_t - B(t,T)}{B(t,T)}$ = loss at the time of default, where w is recovery;
  - $\lambda_t^{\mathbb{P}}$: likelihood of loss *under the actual measure* $\mathbb{P}$;
  - Γ_t : risk premium associated with loss.

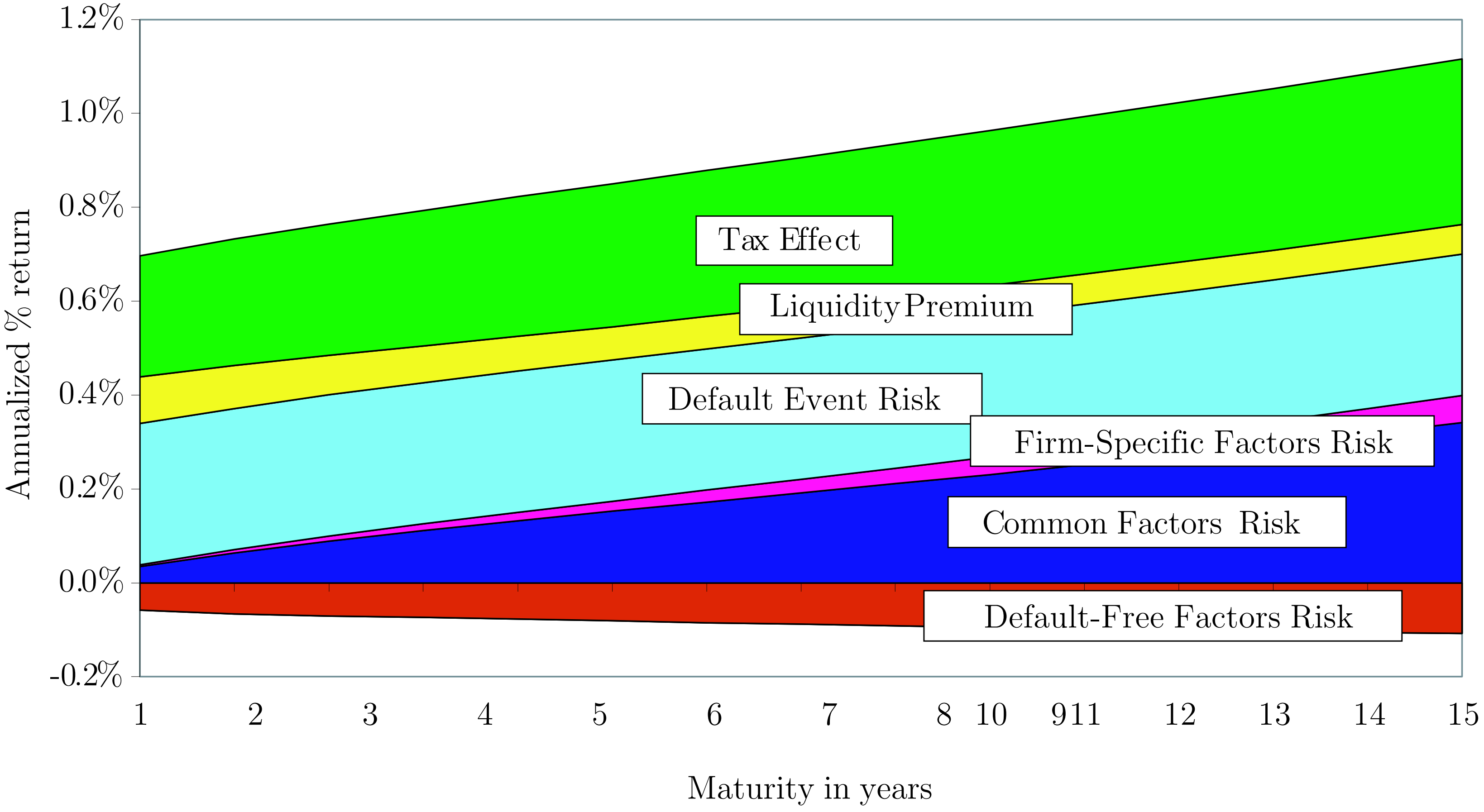


Figure 3: Decomposition of Expected Excess Returns on BBB-rated Bonds. Source: Driessen (2002)

Recovery

- A common practice is to fix recovery $R^{\mathbb{Q}}$ (loss, $L^{\mathbb{Q}}$) at a specific percentage (e.g., 40%) and focus on time-varying $\lambda^{\mathbb{Q}}$.
- For many "linear" instruments (e.g., bonds) trading near par, this approach may be reliable, since it is largely the mean loss rate, $\lambda_t^{\mathbb{Q}} L_t^{\mathbb{Q}}$, that matters.
- However, the valuation of many basket products depends on the distribution of losses.
- The same is true of many option-like products.
- Seasoned CDS positions, trading away from their new-issue spreads, may also be sensitive to recovery assumptions.

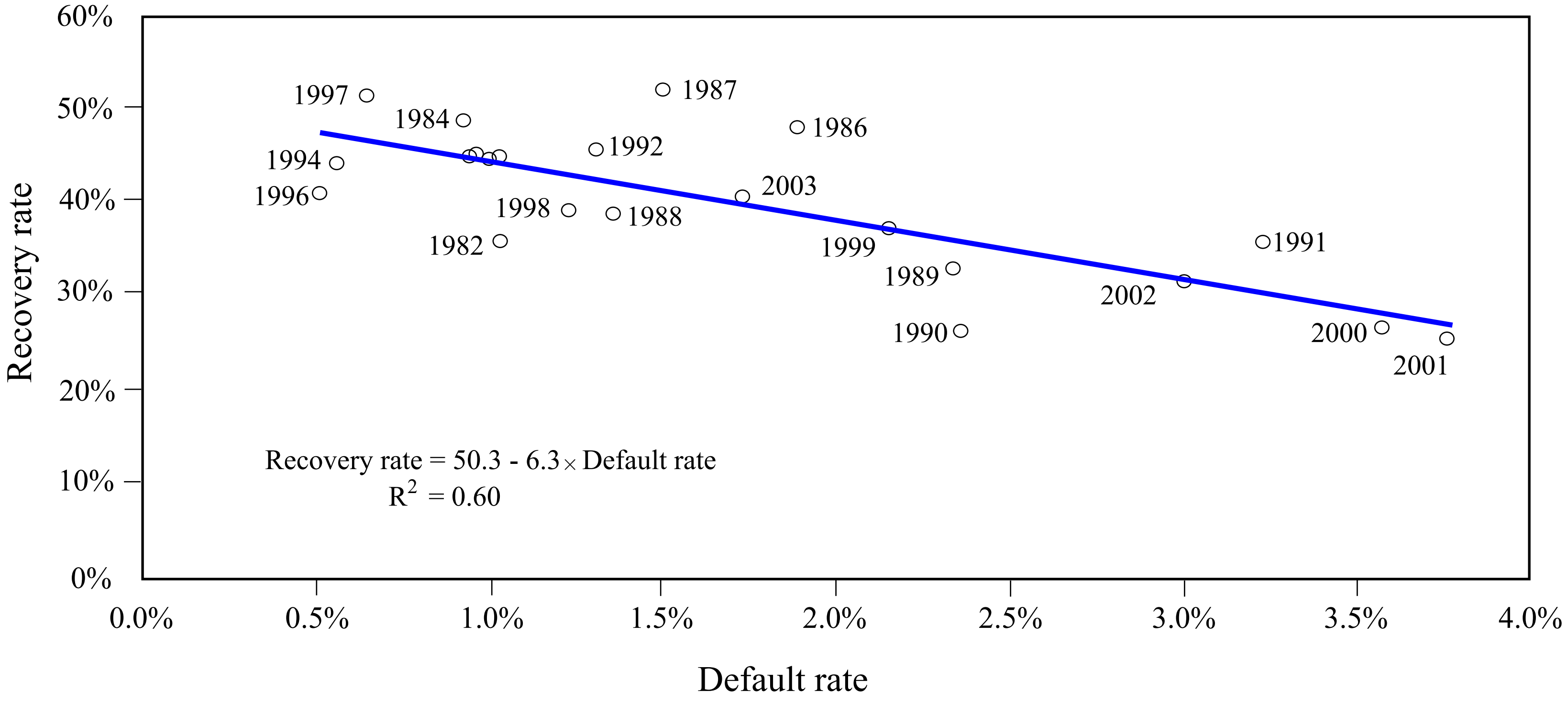


Figure 4: Correlation of Speculative Grade Default and Recovery Rates. Source: Moodys Default and Recovery Report (2004).

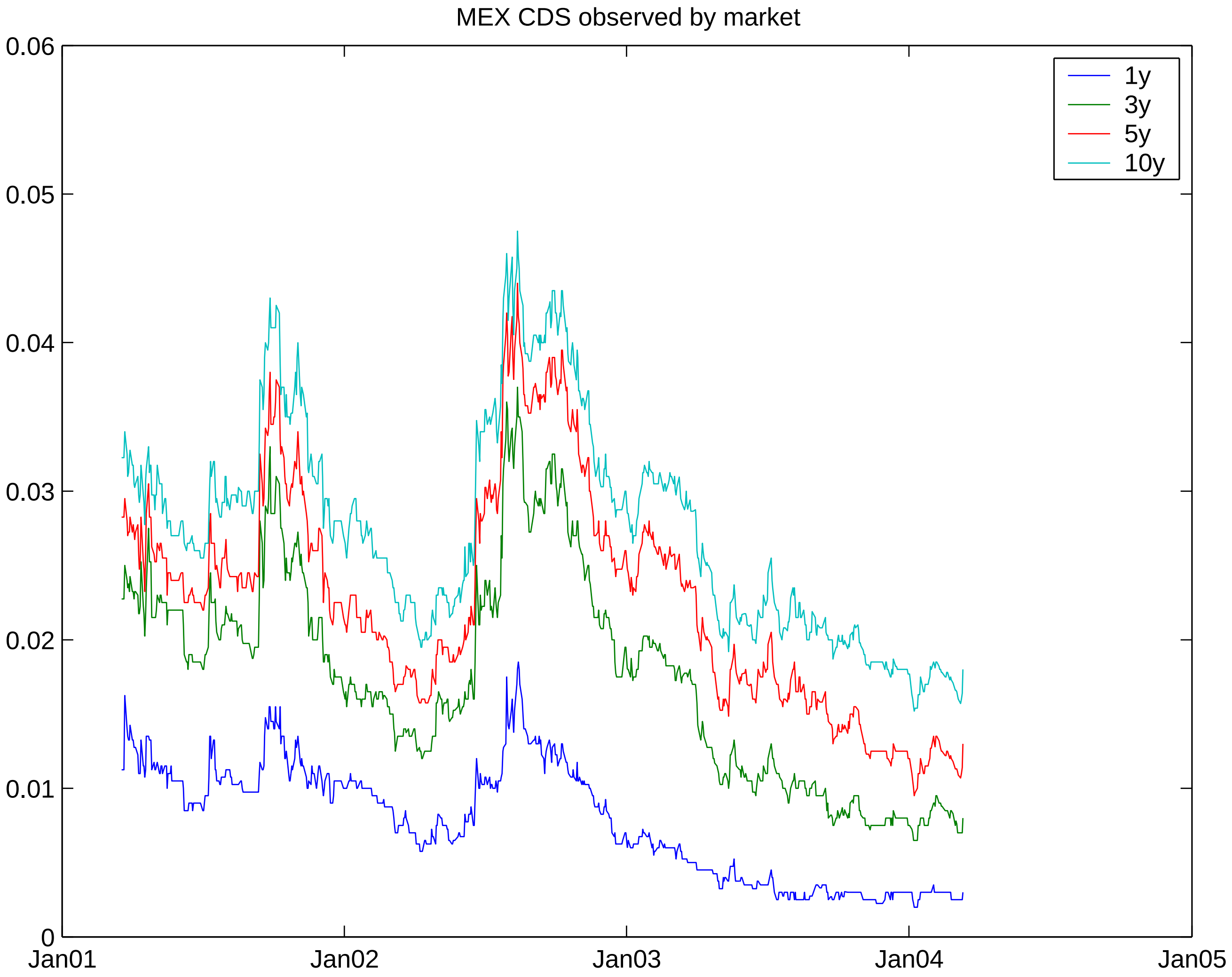


Figure 5: Historical CDS Spreads for Mexico.

- To price these CDS contracts we assumed the riskfree short rate r is

  $$r_t = \rho_0 + X_t^{(1)} + X_t^{(2)}\,,$$

  where, under the risk-neutral dynamics, $X^{(1)}$ and $X^{(2)}$ follow a two-factor ($A_1(2)$) term structure model with stochastic volatility.
- The risk-neutral default intensity is

  $$d\lambda_t^{\mathbb{Q}} = \left(k_3 - K_{33}\lambda_t^{\mathbb{Q}}\right)\,dt + \sigma_3\sqrt{X_t^{(3)}}\,dB_t^{(3)}.$$
- The parameters were estimated by the method of *maximum likelihood* using CDS spreads for maturities 1,3,5, and 10 years.
- Optimum 1: $K_{33} < 0$, explosive! $R^{\mathbb{Q}} = 0.84$.
- Optimum 2: $K_{33} > 0$, stationary. $R^{\mathbb{Q}} = 0.26$.
- Pricing errors are notably smaller under Optimum 1!

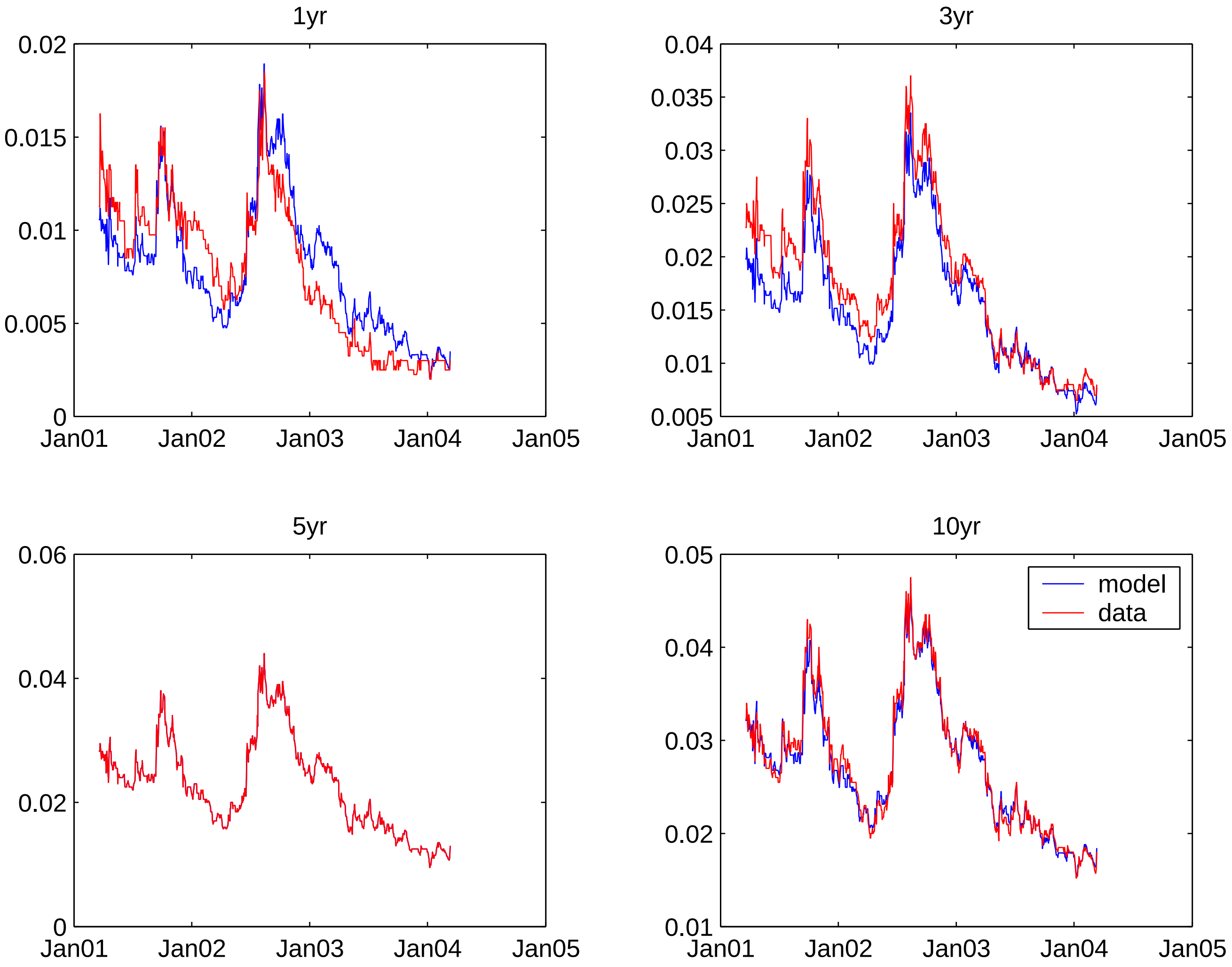


Figure 6: Historical and Fitted CDS Spreads for Mexico.

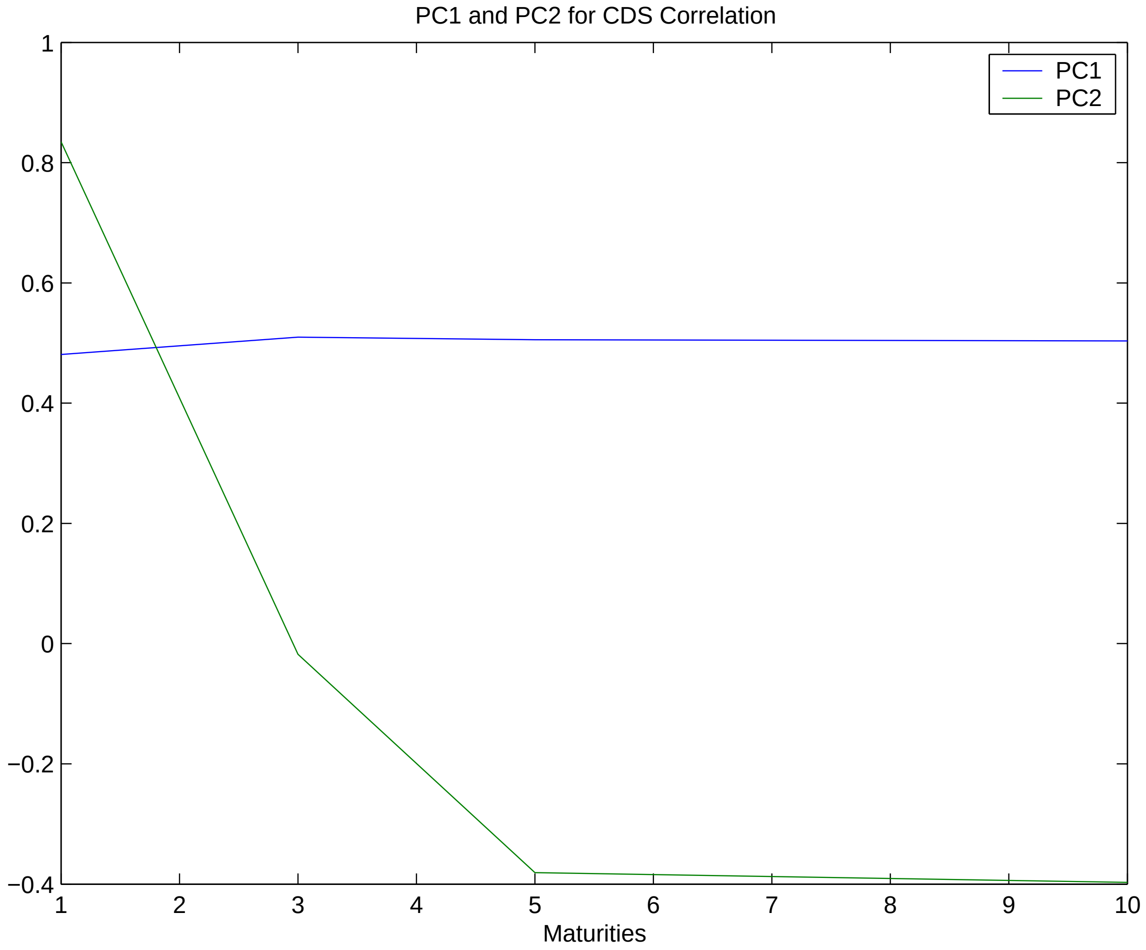


Figure 7: Principal Components Analysis Using Mexican CDS Spreads. PC1 Explains 95% of the variance of CDS spreads.

Risk-Neutral Survival Probabilities

- Using these estimates, we can compute the risk-neutral survival probabilities for different horizons and over time:

$$E_t^{\mathbb{Q}}\left[e^{-\int_t^T \lambda^{\mathbb{Q}}(s)\,ds}\right].$$

- These probabilities were computed for various horizons and over time using the ML estimates from our models.

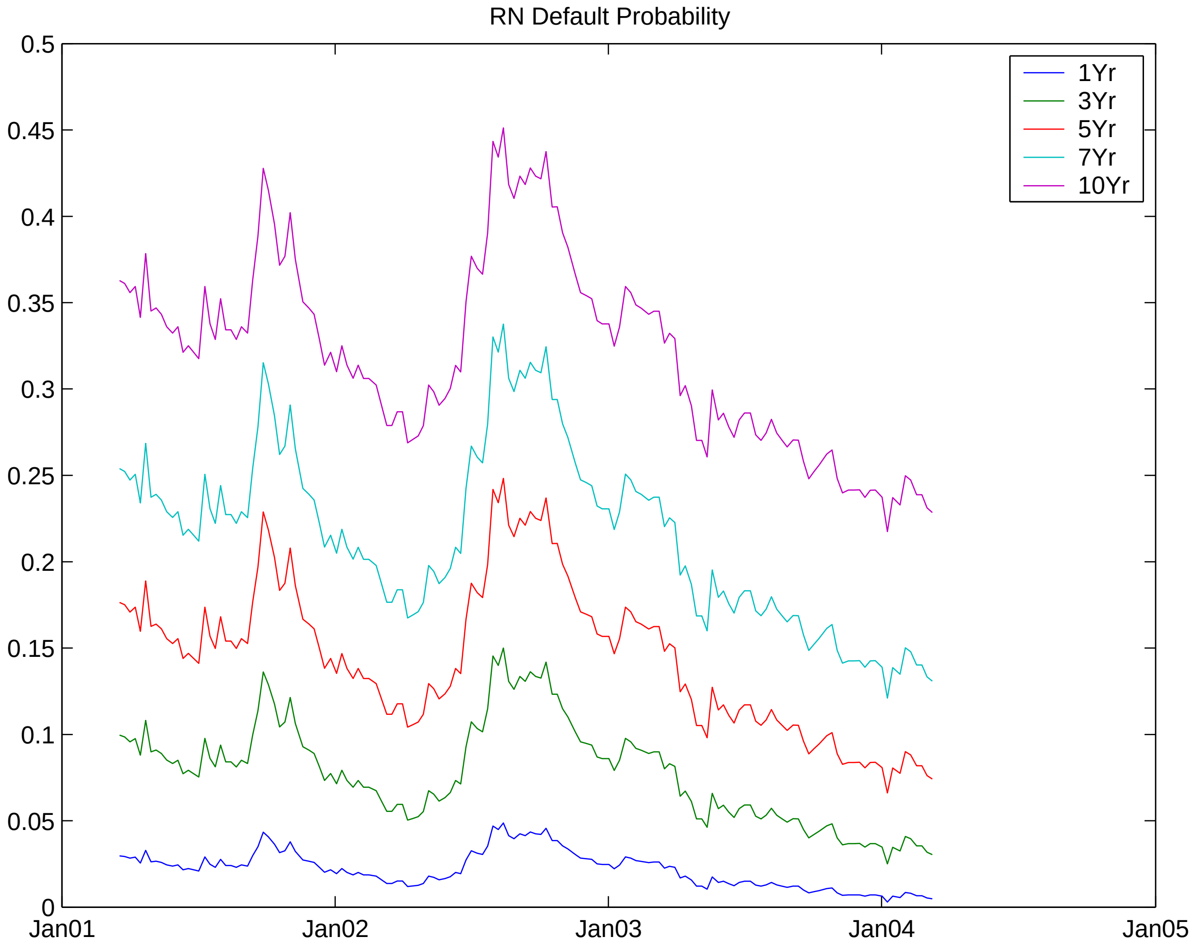


Figure 8: Risk-neutral survival probabilities by horizon; stationary intensity process. $\text{Loss}^{\mathbb{Q}} = 0.74$.

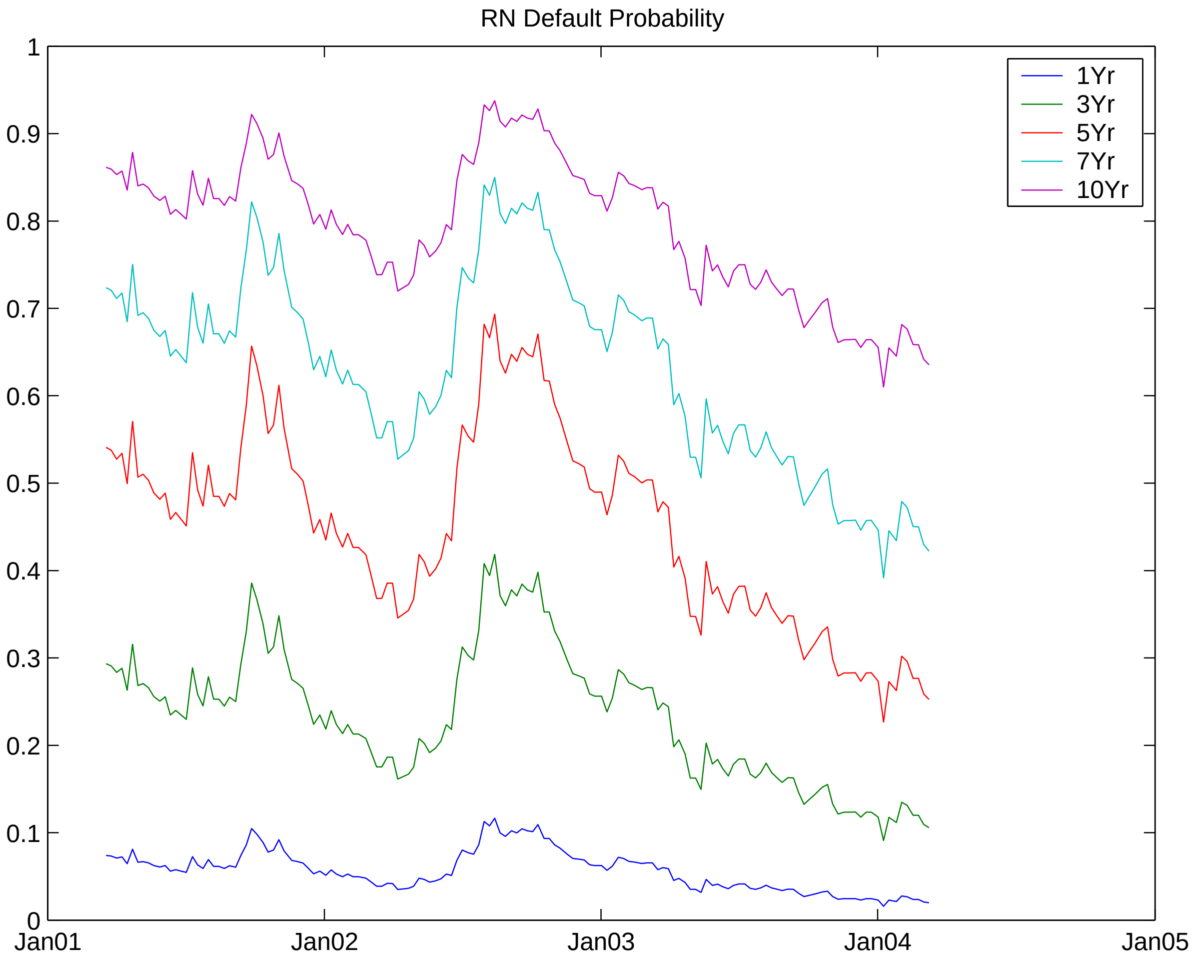


Figure 9: Risk-neutral survival probabilities by horizon; explosive intensity process. $\text{Loss}^{\mathbb{Q}} = 0.16$.