DYNAMIC CONDITIONAL CORRELATION MODELS OF TAIL DEPENDENCE

Robert Engle
NYU Stern

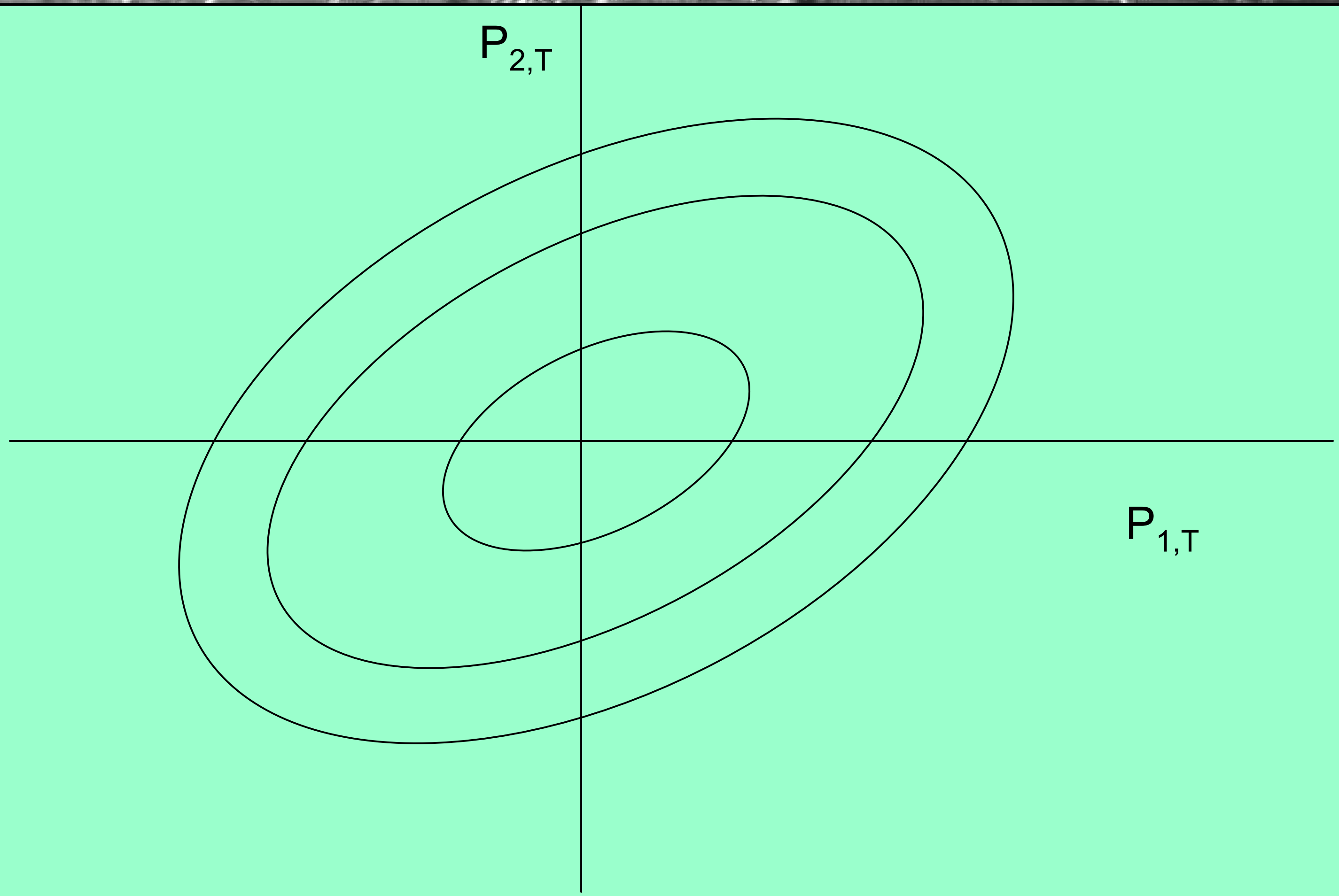
Recent Advances in Credit Risk Research

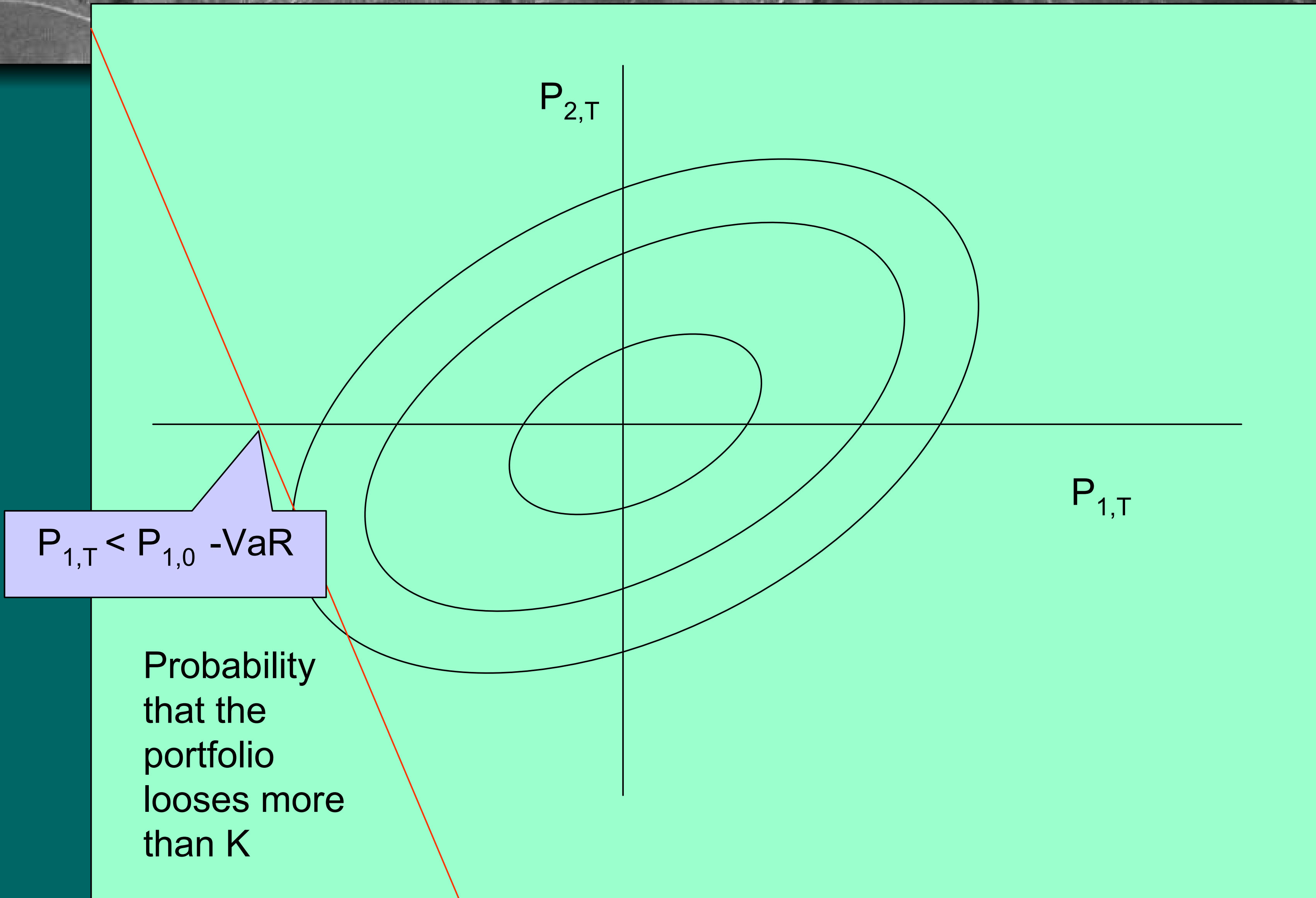
NYU May 2004

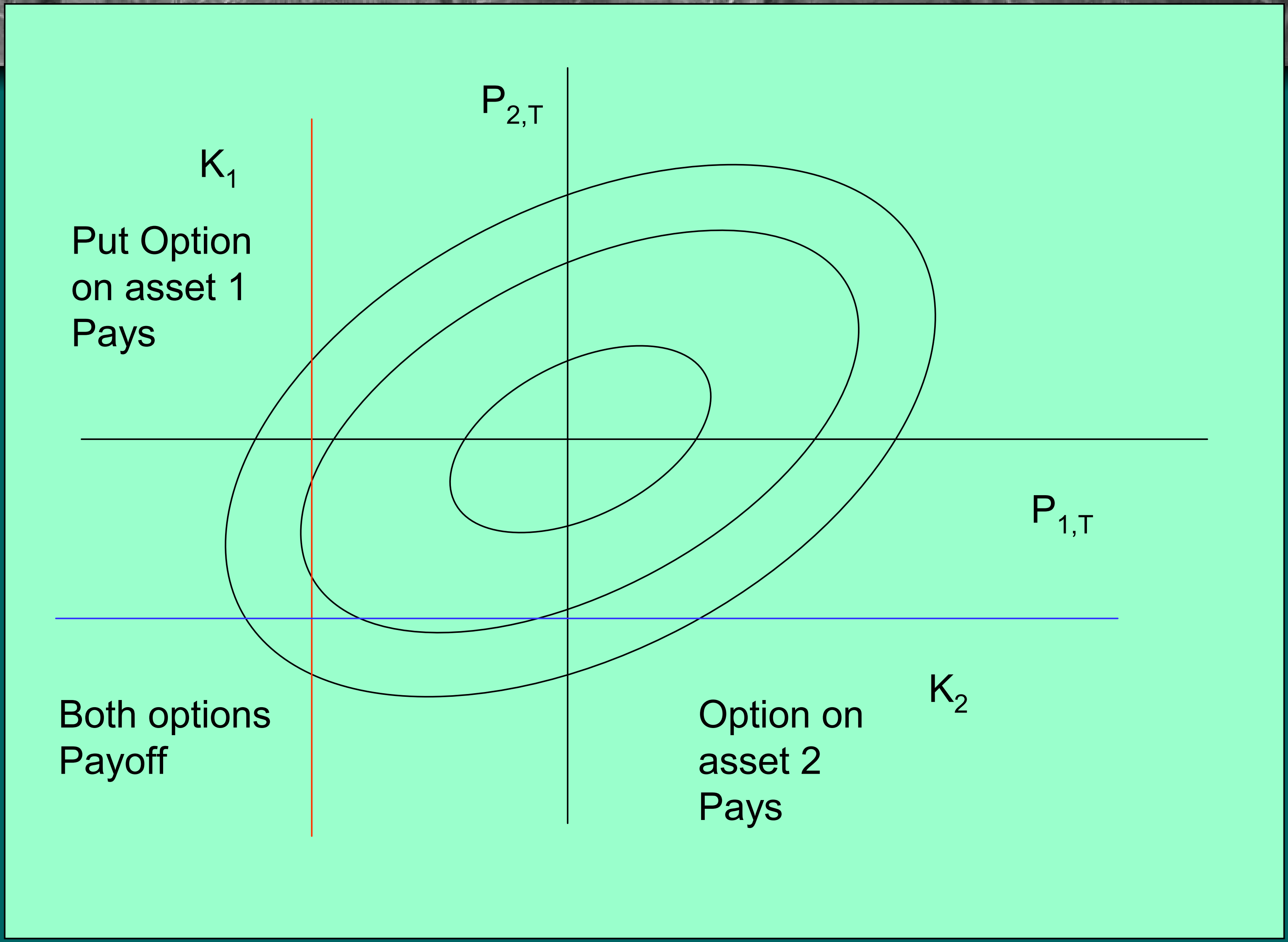
Forecasting the Joint Distribution of Asset Prices

- RISK MANAGEMENT
- OPTIONS PRICING AND CREDIT RISK
- DEFAULT CORRELATIONS AND CREDIT BASKETS, LOAN PORTOLIOS, CDO'S ETC.

Joint Density




$P_{2,T}$
$P_{1,T}$
$P_{1,T} < P_{1,0}$ -VaR
Probability that the portfolio looses more than K



$P_{2,T}$
K_1
Put Option on asset 1 Pays
$P_{1,T}$
K_2
Both options Payoff
Option on asset 2 Pays

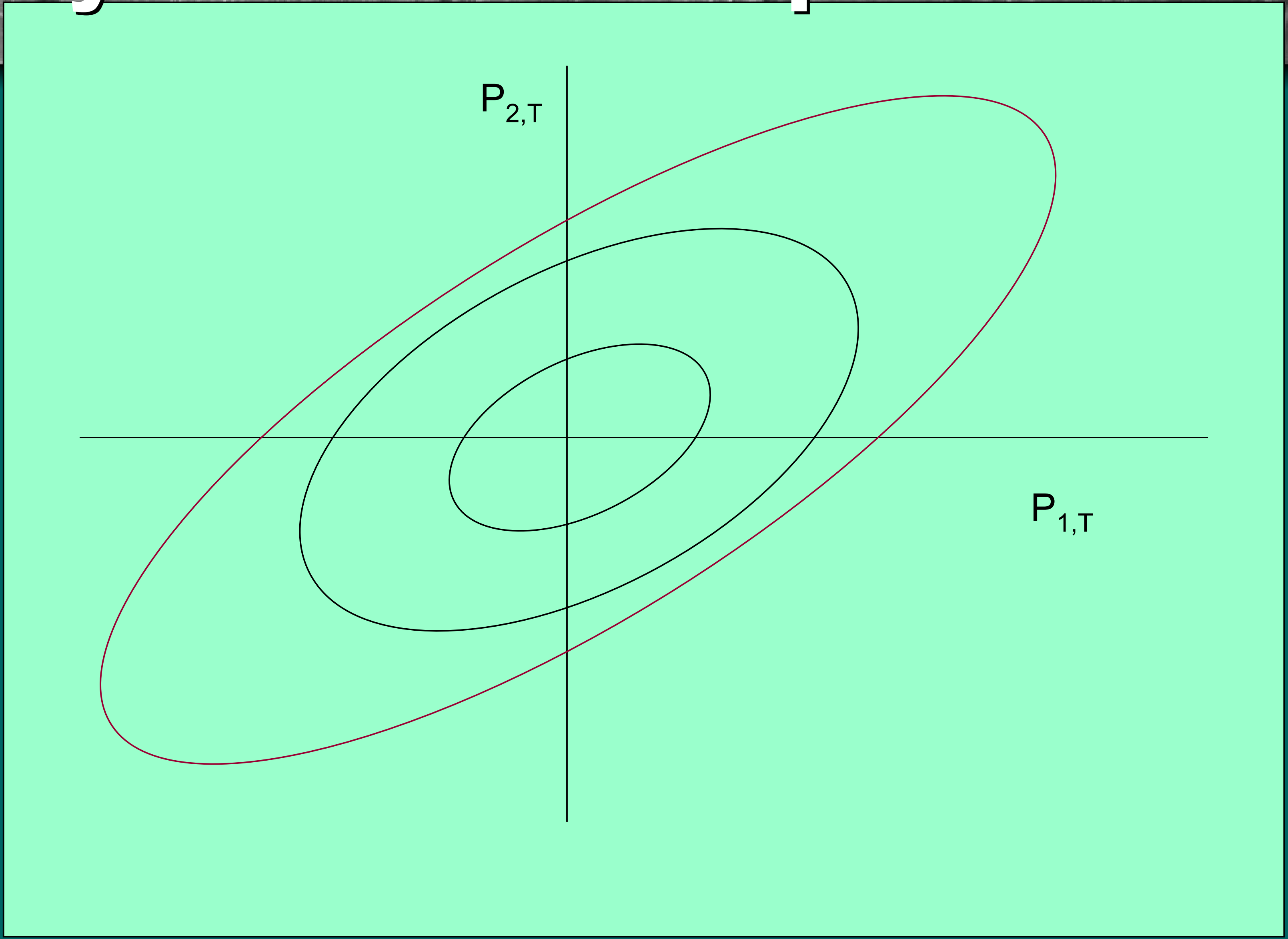
OPTIONS

- Value of each option depends only on the marginal risk neutral distribution
- Correlation between the payoffs depends on the joint distribution.
- Optimal portfolios including options
- Value an option that pays only when both are in the money.

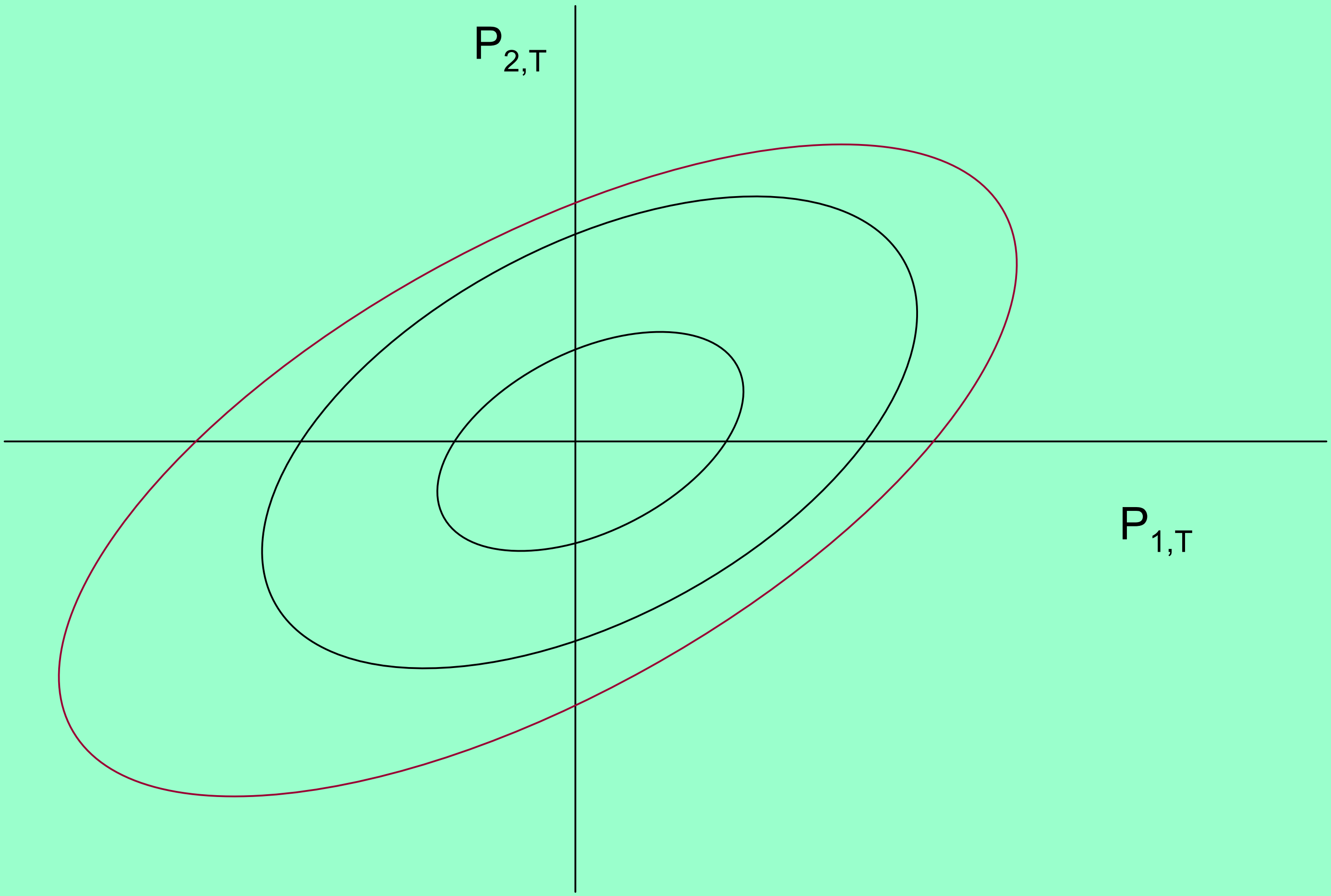
CREDIT RISK

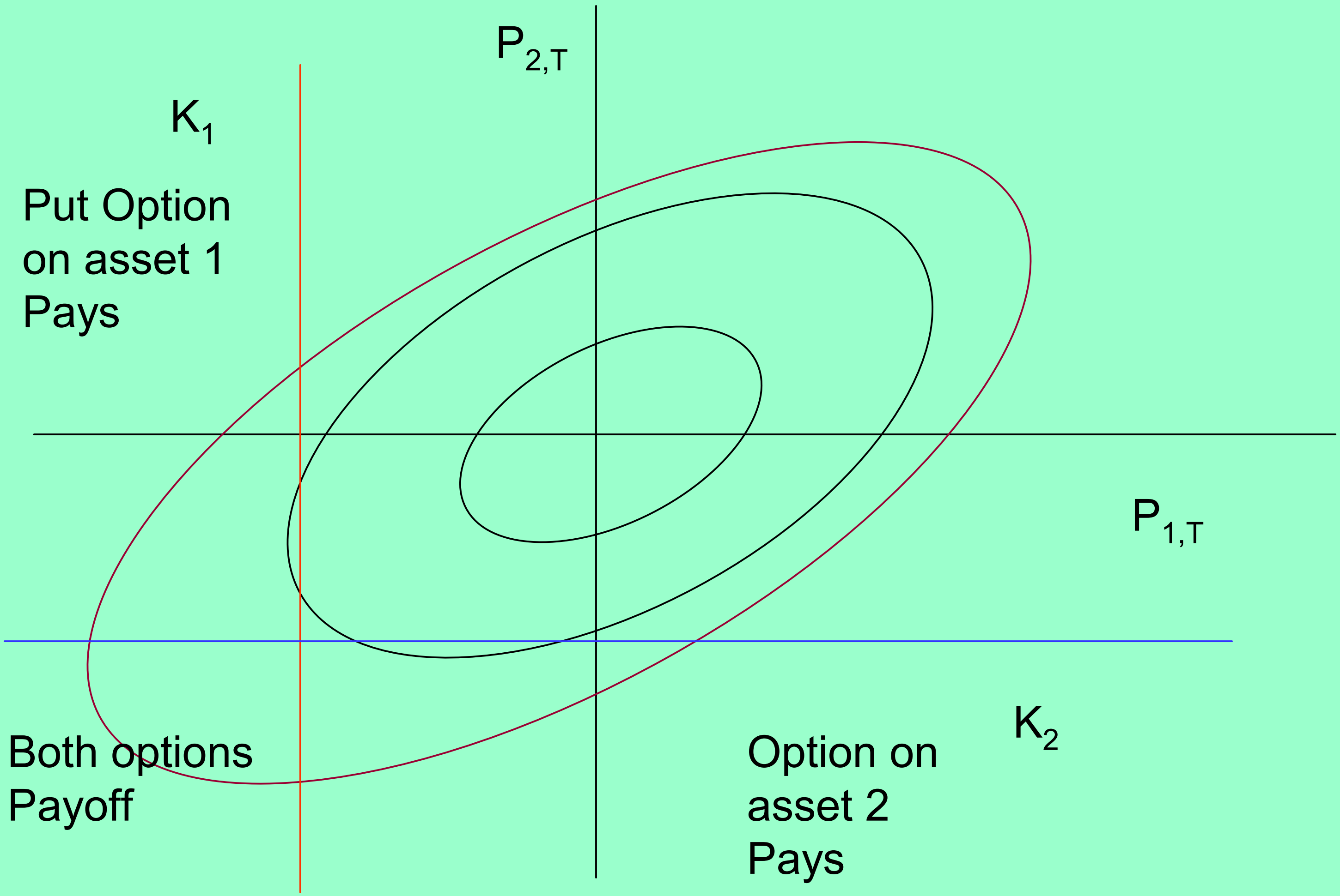
- **Credit Risk** correlation is like this problem where K's are default points and prices are firm values
- **Credit Default Swaps** (CDS) are like options, puts, in firm value.
- **Credit Default Obligations** (CDO) in lower tranches are minima of sums of firm values.
- **First to Default**, Second to Default etc. of Credit Baskets are correlation dependent

Symmetric Tail Dependence



Lower Tail Dependence




$P_{2,T}$
K_1
Put Option
on asset 1
Pays
$P_{1,T}$
K_2
Both options
Payoff
Option on
asset 2
Pays

JOINT DISTRIBUTIONS

- For a vector of kx1 random variables Y with cumulative distribution function F

$$F\left(y_1,\ldots,y_k\right)=P\left(Y_1<y_1,\ldots,Y_k<y_k\right)$$

- Assuming for simplicity that it is continuously differentiable, then the density function is:

$$f\left(y_1,\ldots,y_k\right)=\frac{\partial^k F\left(y\right)}{\partial y_1\ldots\partial y_k}$$

UNIVARIATE PROPERTIES

- For any joint distribution function F, there are univariate distributions F_i and densities f_i defined by:

$$F_i(y_i) = P(Y_i < y_i) = F(\infty,\ldots,\infty, y_i, \infty,\ldots,\infty)$$

$$f_i(y_i) = \frac{\partial F_i}{\partial y_i}$$

- $U_i = F_i(Y_i)$ is a uniform random variable on the interval (0,1)
- What is the joint distribution of

$$U = (U_1,\ldots,U_k)$$

COPULA

- The joint distribution of these uniform random variables is called a copula;
  - it only depends on ranks and
  - is invariant to monotonic transformations.

$$U = \left(U_1, \ldots, U_k\right) \sim C\left(u_1, .., u_k\right)$$

- Equivalently

$$F\left(y\right) = C\left(F_1\left(y_1\right), \ldots, F_k\left(y_k\right)\right)$$

$$C\left(u\right) = F\left(F_1^{-1}\left(u_1\right), \ldots, F_k^{-1}\left(u_k\right)\right)$$

COPULA DENSITY

- Again assuming continuous differentiability, the copula density is

$$c(u) = \frac{\partial^k C(u)}{\partial u_1, \ldots \partial u_k}$$

- From the chain rule or change of variable rule, the joint density is the product of the copula density and the marginal densities

$$f(y) = c(u) f_1(y_1) f_2(y_2) \ldots f_k(y_k)$$

Tail Dependence

Upper and lower tail dependence:

$$\lim_{\theta \to 1} \Pr\left(r_{1,T} > K_{1,\theta} \middle| r_{2,T} > K_{2,\theta}\right) = \lambda_U$$

$$\lim_{\theta \to 0} \Pr\left(r_{1,T} < K_{1,\theta} \middle| r_{2,T} < K_{2,\theta}\right) = \lambda_L$$

$$\Pr\left(r_{1,T} < K_{1,\theta}\right) = \Pr\left(r_{2,T} < K_{2,\theta}\right) \equiv \theta$$

- **These depend only on the copula**
- **For a joint normal, these are both zero!**

DEFAULT CORRELATIONS

- Let I_i be the event that firm i defaults,
- Then the default correlation is the correlation between I_1 and I_2 which can be computed conditional on today's information set.
- If the probability of default for each firm is θ , then:

$$\rho_{1,2}(\theta) = \frac{\Pr(I_1 * I_2 = 1) - \theta^2}{\theta(1-\theta)}$$

Default Correlations and Tail Dependence

- When defaults are unlikely, these are related to the tail dependence measure
- Take the limit as θ becomes small

$$\lim_{\theta \to 0} \rho_{1,2}(\theta) = \frac{\lambda_L - \theta}{(1-\theta)}$$

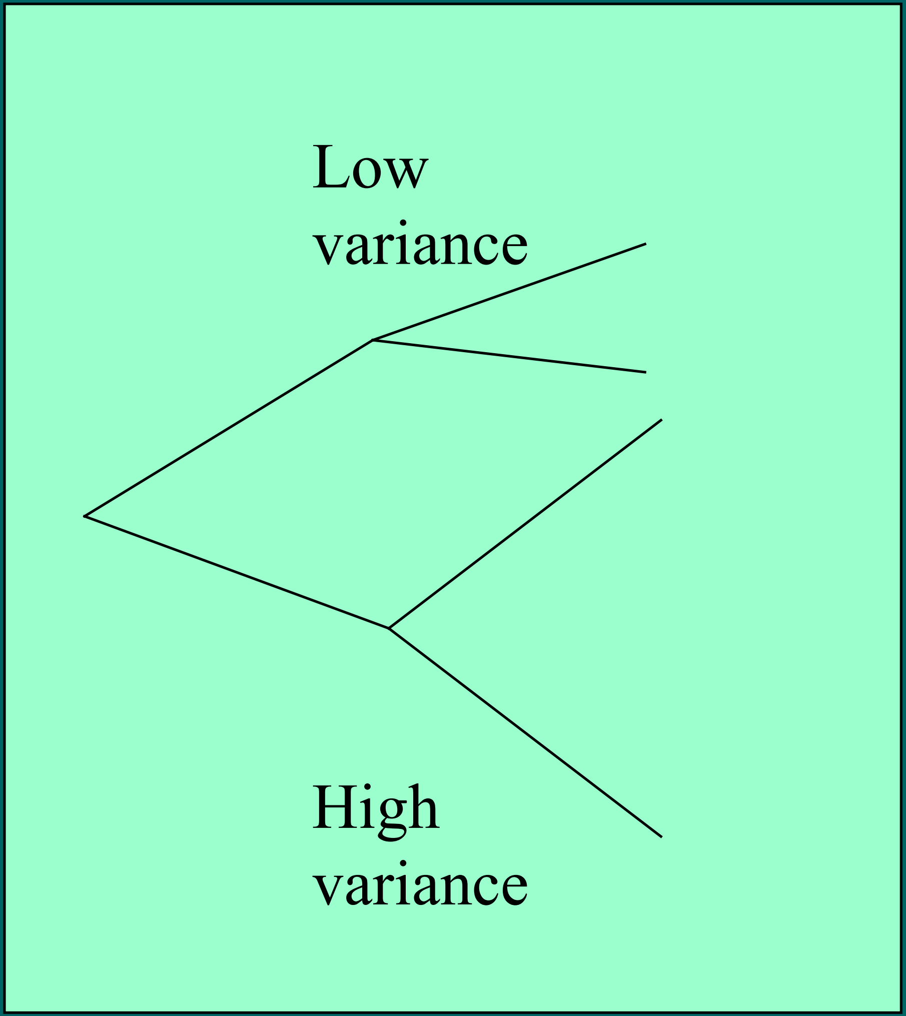
  - Under normality or independence, the limiting default correlation is zero
  - Under lower tail dependence it is positive.

DYNAMIC CORRELATIONS

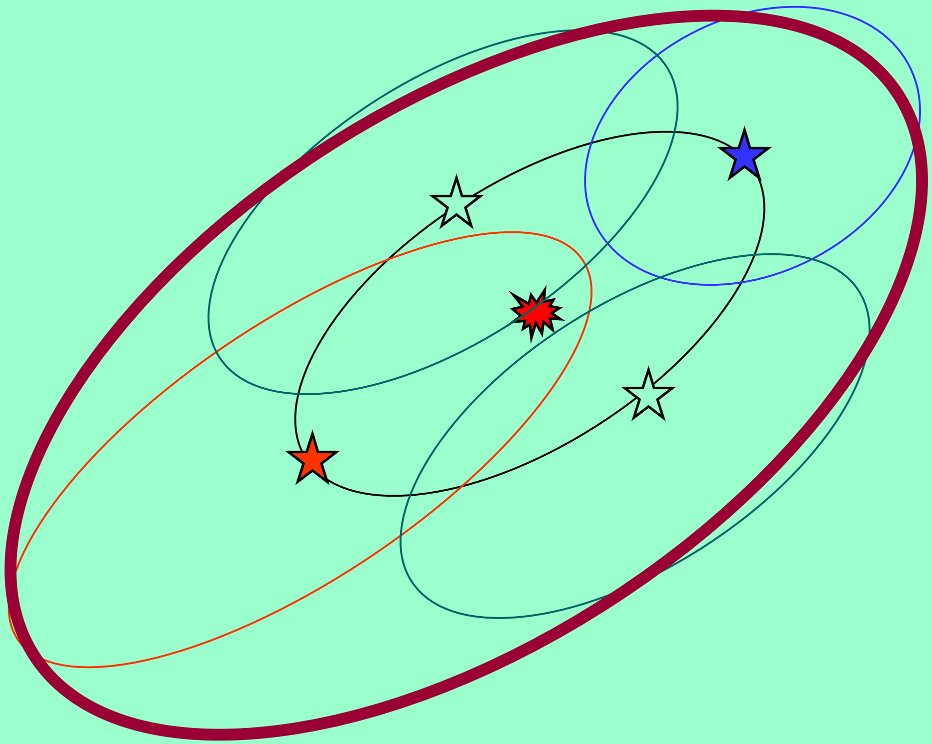
- A joint distribution can be defined for any horizon. Long horizon distributions can be built up from short horizons
- Multivariate GARCH gives many possible models for daily correlations. The implied multi-period distribution will generally show symmetric tail dependence
- Special asymmetric multivariate GARCH models give greater lower tail dependence.

TWO PERIOD RETURNS

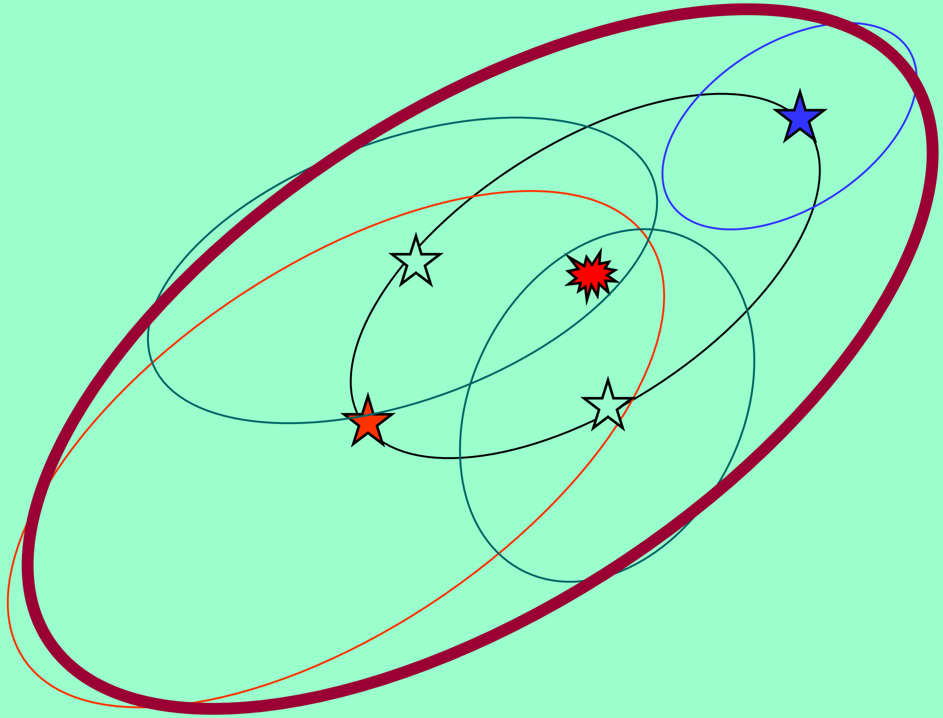
- Two period return is the sum of two one period continuously compounded returns
- Look at binomial tree version
- Asymmetry gives negative skewness



Two period Joint Returns with Constant Volatilities



Two period Joint Returns with Constant Correlations



TERM STRUCTURE OF DEFAULT CORRELATIONS

- **SIMULATE FIRM VALUES, AND CALCULATE DEFAULTS**
- **VARIOUS ASSUMPTIONS CAN BE MADE.**
  - First passage defines default
  - Value at end of period defines default status
  - Default frontier can be constant or changing
- **LOSS COULD BE CONSTANT OR STATE DEPENDENT**

UPDATING

- As each day passes, the remaining time before maturity of a credit derivative will be shorter and the joint distribution of the outcome will have changed.
- How do you update this distribution?
- How do you hedge your position?

WHAT COPULA TO USE?

- Rather than select a copula based on derivative prices
- Fit a Multivariate Volatility model to equity or asset returns.
- The time aggregated value is the joint distribution and defines the copula
- Simulation is an easy way to price, hedge and update positions.

Dynamic Conditional Correlation

- DCC is a new type of multivariate GARCH model that is particularly convenient for big systems. See Engle(2002) or Engle(2004).

Dynamic Conditional Correlation

- DCC model is a new type of multivariate GARCH model that is particularly convenient for big systems. See Engle(2002) or Engle(2004)
- Motivation: the conditional correlation of two returns with mean zero is:

$$\rho_t = \frac{E_{t-1}\left(r_{1,t} r_{2,t}\right)}{\sqrt{E_{t-1}\left(r_{1,t}^2\right) E_{t-1}\left(r_{2,t}^2\right)}}$$

DCC

- **Then defining the conditional variance and standardized residual as**

$$h_{i,t} \equiv E_{t-1}\left(r_{i,t}^2\right),\ and\ \ \varepsilon_{i,t} = r_{i,t} / \sqrt{h_{i,t}}$$

- **All the volatilities cancel, giving**

$$\rho_t = \frac{E_{t-1}\left(\varepsilon_{1,t}\varepsilon_{2,t}\right)}{\sqrt{E_{t-1}\left(\varepsilon_{1,t}^2\right)E_{t-1}\left(\varepsilon_{2,t}^2\right)}} = E_{t-1}\left(\varepsilon_{1,t}\varepsilon_{2,t}\right)$$

DCC

1. Estimate volatilities for each asset and compute the *standardized residuals* or *de-volatilized returns*.
2. Estimate the time varying covariances between these using a maximum likelihood criterion and one of several models for the correlations.
3. Form the correlation matrix guaranteed to be positive definite.

HOW IT WORKS

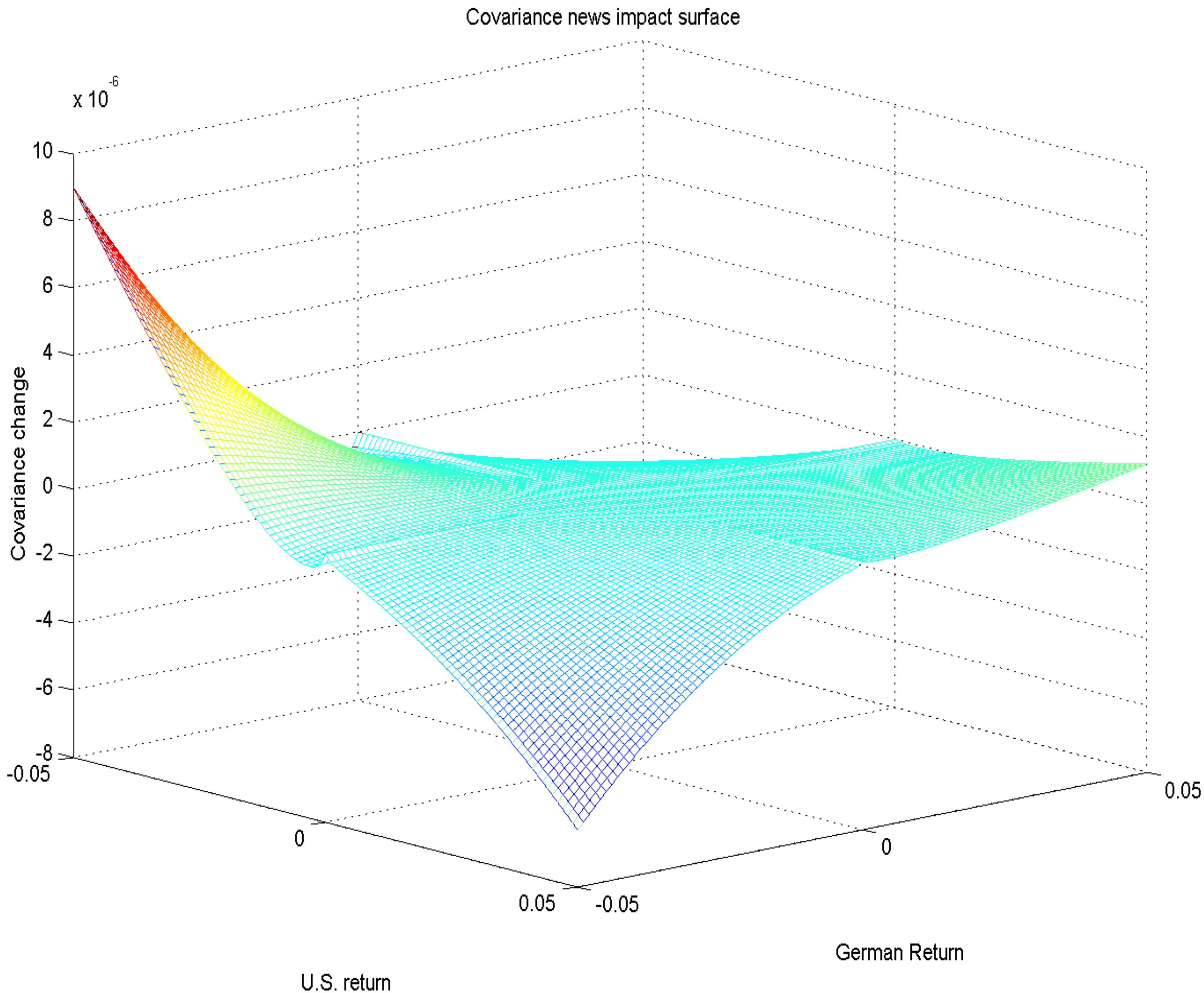
- When two assets move in the same direction, the correlation is increased slightly.
- When they move in the opposite direction it is decreased.
- This effect may be stronger in down markets.
- The correlations often are assumed to only temporarily deviate from a long run mean

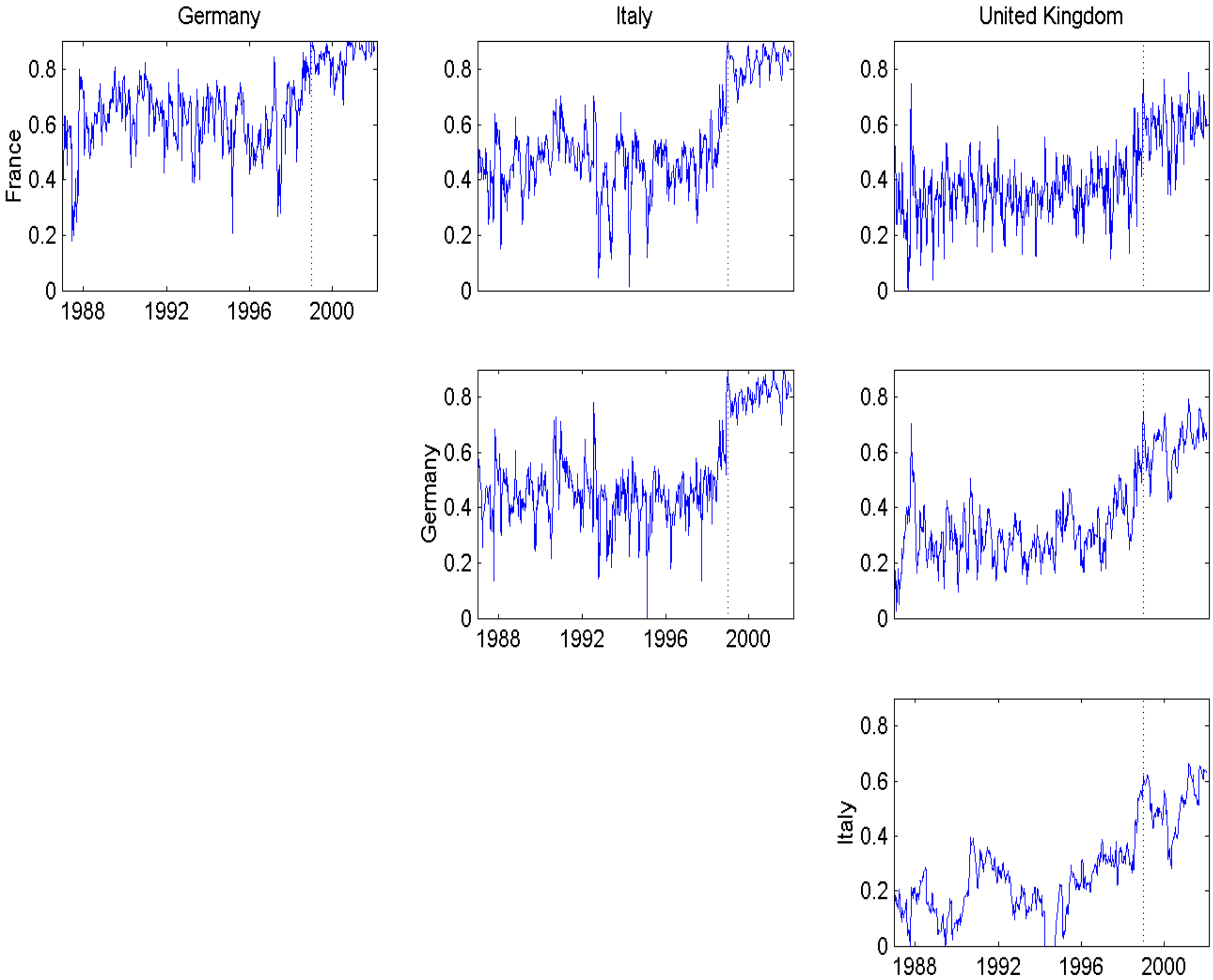
Data

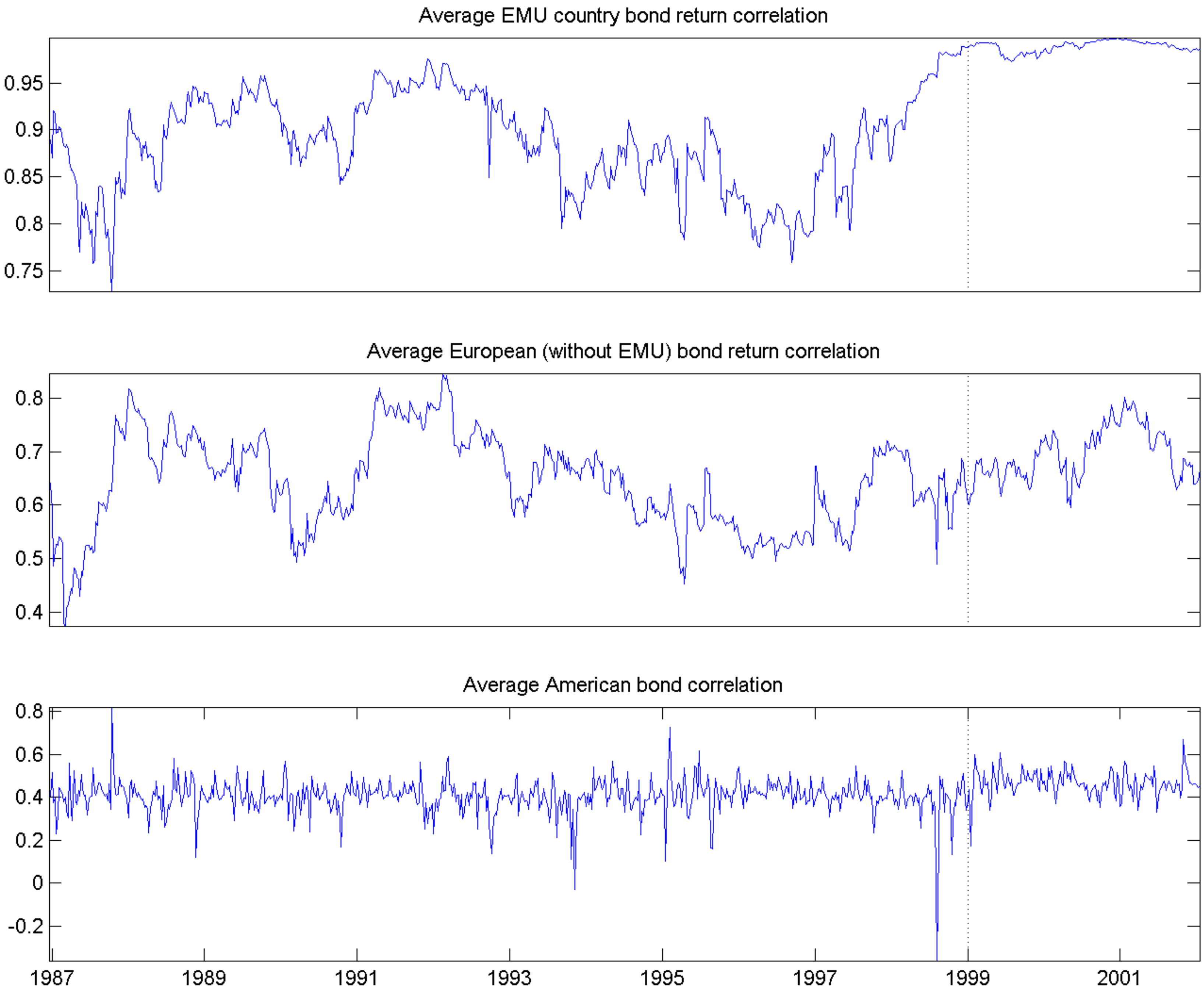
- Weekly $ returns Jan 1987 to Feb 2002 (785 observations)
- 21 Country Equity Series from FTSE All-World Index
- 13 Datastream Benchmark Bond Indices with 5 years average maturity

GARCH Models
(asymmetric in orange)

GARCH	3EQ,8BOND
AVGARCH	0
NGARCH	1BOND
EGARCH	6EQ,1BOND
ZGARCH	8EQ,1BOND
GJR-GARCH	3EQ,1BOND
APARCH	0
AGARCH	1EQ,1BOND
NAGARCH	0


Covariance news impact surface
x 10^-6
10
8
6
4
2
0
-2
-4
-6
-8
Covariance change
-0.05
0
0.05
U.S. return
-0.05
0
0.05
German Return


Germany
Italy
United Kingdom
France
Germany
Italy
0.8
0.6
0.4
0.2
0
1988
1992
1996
2000


Average EMU country bond return correlation
0.95
0.9
0.85
0.8
0.75
Average European (without EMU) bond return correlation
0.8
0.7
0.6
0.5
0.4
Average American bond correlation
0.8
0.6
0.4
0.2
0
-0.2
1987
1989
1991
1993
1995
1997
1999
2001

EXPLANATIONS?

- Euro zone?
- Globalization?
- Market Declines?
- Market Volatility?
- ALL?

REFERENCES

- Engle, 2002, Dynamic Conditional Correlation-A Simple Class of Multivariate GARCH Models, *Journal of Business and Economic Statistics*
  - *Bivariate examples and Monte Carlo*
- Engle and Sheppard, 2002 "Theoretical and Empirical Properties of Dynamic Conditional Correlation Multivariate GARCH", NBER Discussion Paper, and UCSD DP.
  - Models of 30 Dow Stocks and 100 S&P Sectors
- Cappiello, Engle and Sheppard, 2002, "Asymmetric Dynamics in the Correlations of International Equity and Bond Returns", UCSD Discussion Paper
  - Correlations between 34 International equity and bond indices

CONCLUSIONS

- Dynamic Correlation models give a flexible strategy for modeling non-normal joint density functions or copulas.
- Updating can be used to re-price or re-hedge positions
- The model can be of equity values or firm values and risk neutral or empirical measures, depending on the application.